Algebraic properties of L-fuzzy approximation operators on residuated lattices

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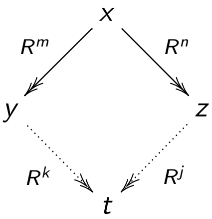
We consider properties of L-fuzzy relations on a residuated lattice L and show a characterization theorem of confluent L-fuzzy relation:

$$R^m(x,y) \odot R^n(x,z) \leq \bigvee_{t \in U} (R^k(y,t) \odot R^j(z,t)) \quad (m,n,k,j \in \mathbb{N}),$$



R is confluent $\iff$ $\overline{\mathrm{R}}^n \, \underline{\mathrm{R}}^j \leq \underline{\mathrm{R}}^m \, \overline{\mathrm{R}}^k$.

residuated lattice

Let $\mathcal{L} =< L, \wedge, \vee, \odot, 0, 1 >$ be a residuated lattice, i.e.,

(i) $< L, \wedge, \vee, 0, 1 >$ is a bounded lattice;

(ii) $< L, \odot, 1 >$ is a commutative monoid;

(iii) For all $a, b, c \in L$,

$$a \odot b \leq c \Longleftrightarrow a \leq b \to c.$$

Proposition 1.([1]) For all $a, b, c, a_i, b_i \in L$, we have

(1) $a \odot a' = 0$, where $a' = a \rightarrow 0$;

(2) $a \leq b \iff a \rightarrow b = 1$;

(3) $a \odot (a \rightarrow b) \leq b$;

(4) $a \leq b \implies a \odot c \leq b \odot c, c \rightarrow a \leq c \rightarrow b,$
$b \rightarrow c \leq a \rightarrow c$;

(5) $1 \rightarrow a = a$;

(6) $a \vee (b \rightarrow c) \leq b \rightarrow a \vee c$;

L-fuzzy relation

Let U be a non-empty set and L a residuated lattice. A map $R : U \times U \to L$ is called an *L-fuzzy relation* on U.
There are two special L-fuzzy relations ω, ι on U defined by

$$\omega(x, y) \stackrel{\text{def}}{=} \begin{cases} 1 & (x = y) \\ 0 & (x \neq y), \end{cases}$$

$$\iota(x, y) \stackrel{\text{def}}{=} 1 \quad (\forall, x, y \in U).$$

We define an order $\leq$ on the set $\mathcal{R}(U) = L^{U \times U}$ of all L-fuzzy relations on U as usual: For $R, S \in \mathcal{R}(U)$,

$$R \leq S \iff R(x,y) \leq S(x,y) \quad (\forall\, x, y \in U).$$

For $R, S \in \mathcal{R}(U)$, we define operations $^{-1}$ and $\circ$

$$R^{-1}(x,y) = R(y,x) \quad (\forall\, x, y \in U);$$

$$(R \circ S)(x,y) = \bigvee_{z \in X} (R(x,z) \odot S(z,y)) \quad (\forall\, x, y \in U).$$

Proposition 2. For any $R, S \in \mathcal{R}(U)$,

(1) $R^{-1}, R \circ S \in \mathcal{R}(U)$;

(2) $R \leq S \iff R^{-1} \leq S^{-1}$;

(3) $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$;

(4) $(R^{-1})^{-1} = R$.

For a non-empty set U and an L-fuzzy relation R on U, a structure (U, R) is called an *L-fuzzy approximation space*. We define an *upper (lower) L-fuzzy approximation operators* $\overline{\mathrm{R}}$ $(\underline{\mathrm{R}}) : L^U \to L^U$ as follows: For all $A \in L^U$,

$$\overline{\mathrm{R}}(A)(x) \overset{\text{def}}{=} \bigvee_{y \in U} (R(x, y) \odot A(y))$$

$$\underline{\mathrm{R}}(A)(x) \overset{\text{def}}{=} \bigwedge_{y \in U} (R(x, y) \to A(y))$$

An order $\leq$ is defined on the set of all L-fuzzy approximation operators on U as usual: For $F, G : L^U \to L^U$,

$$F \leq G \iff F(A)(x) \leq G(A)(x) \;\; (\forall\, x \in U,\; A \in L^U).$$

Proposition 3. For any L-fuzzy relation $R, R_i, S, S_i \in L^{U\times U}$,

(1) $\overline{\mathrm{R}}, \underline{\mathrm{R}}$ are order-preserving, that is,

$$A \leq B \;\Rightarrow\; \overline{\mathrm{R}}(A) \leq \overline{\mathrm{R}}(B),\; \underline{\mathrm{R}}(A) \leq \underline{\mathrm{R}}(B);$$

(2) If $\overline{\mathrm{R_i}} \leq \overline{\mathrm{S_i}}$ $\;(i = 1, 2)$, then $\overline{\mathrm{R_1}}\,\overline{\mathrm{R_2}} \leq \overline{\mathrm{S_1}}\,\overline{\mathrm{S_2}}$;

(3) If $\underline{\mathrm{R_i}} \leq \underline{\mathrm{S_i}}$ $\;(i = 1, 2)$, then $\underline{\mathrm{R_1}}\,\underline{\mathrm{R_2}} \leq \underline{\mathrm{S_1}}\,\underline{\mathrm{S_2}}$.

Proposition 4. For any L-fuzzy relation R on U and $A, B \in L^U$, we have

$$\overline{\mathrm{R}}(A) \leq B \;\Leftrightarrow\; A \leq \underline{\mathrm{R}^{-1}}(B) \qquad (i.e., \; \overline{\mathrm{R}} \dashv \underline{\mathrm{R}^{-1}});$$
$$\overline{\mathrm{R}^{-1}}(A) \leq B \;\Leftrightarrow\; A \leq \underline{\mathrm{R}}(B) \qquad (i.e., \; \overline{\mathrm{R}^{-1}} \dashv \underline{\mathrm{R}}).$$

$$\Downarrow$$

$$\overline{\mathrm{R}}\,\underline{\mathrm{R}^{-1}} \leq \mathrm{I} \leq \underline{\mathrm{R}^{-1}}\,\overline{\mathrm{R}},$$

where I is defined by $\mathrm{I}(A) = A$ for all $A \in L^U$

Proposition 5. For all L-fuzzy relation $R, S \in \mathcal{R}(U)$, we have

(1) $\overline{\mathrm{R}}\,\mathrm{I} = \mathrm{I}\,\overline{\mathrm{R}} = \overline{\mathrm{R}}$;

(2) $\overline{(R \circ S)} = \overline{\mathrm{R}}\,\overline{\mathrm{S}}$;

(3) $\underline{(R \circ S)} = \underline{\mathrm{R}}\,\underline{\mathrm{S}}$;

(4) $R \leq S \iff \overline{\mathrm{R}} \leq \overline{\mathrm{S}}$;

(5) $\overline{\mathrm{R}} \leq \overline{\mathrm{S}} \iff \underline{\mathrm{S}^{-1}} \leq \underline{\mathrm{R}^{-1}}$, hence,

$$R \leq S \iff \overline{\mathrm{R}} \leq \overline{\mathrm{S}} \iff \underline{\mathrm{S}^{-1}} \leq \underline{\mathrm{R}^{-1}}.$$

Proof We only show (4) : $R \leq S \iff \overline{\mathrm{R}} \leq \overline{\mathrm{S}}$.

($\Rightarrow$) If $R \leq S$, since

$$\overline{\mathrm{R}}(A)(x) = \bigvee_y (R(x,y) \odot (A(y)) \leq \bigvee_y (S(x,y) \odot A(y)) = \overline{\mathrm{S}}(A)(x)$$

for all $x \in U, A \in L^U$, we have $\overline{\mathrm{R}} \leq \overline{\mathrm{S}}$.

($\Leftarrow$) Conversely, suppose that $\overline{\mathrm{R}} \leq \overline{\mathrm{S}}$. Then we get

$$R(x,y) = \overline{\mathrm{R}}(\mathbf{1}_y)(x) \leq \overline{\mathrm{S}}(\mathbf{1}_y)(x) = S(x,y) \quad (\forall x, y \in U),$$

hence $R \leq S$. $\square$

Corollary 1. For all L-fuzzy relations R, S on U, the following are equivalent:

$$(1)\ R \leq S \qquad (2)\ \overline{\mathrm{R}} \leq \overline{\mathrm{S}} \qquad (3)\ \underline{\mathrm{S}} \leq \underline{\mathrm{R}}.$$

For all order-preserving operators $\mathrm{F}, \mathrm{G} : L^U \to L^U$, it is easy to show that

(1) The product operator $\mathrm{F}\,\mathrm{G}$ *of F and G is also an order preserving operator;*

(2) For any operator $\mathrm{H} : L^U \to L^U$*, we have*

$$\mathrm{F} \le \mathrm{G} \Longrightarrow \mathrm{F}\,\mathrm{H} \le \mathrm{G}\,\mathrm{H},\ \ \mathrm{H}\,\mathrm{F} \le \mathrm{H}\,\mathrm{G}.$$

$$\Downarrow$$

Finite product of operators $\overline{\mathrm{R}}, \overline{\mathrm{R}^{-1}}, \underline{\mathrm{R}}, \underline{\mathrm{R}^{-1}}$ is order-preserving.

Proposition 6. Let α, β be finite relational products of R, R^{-1}. Then we have

$$\overline{\alpha} \le \overline{\beta} \iff \underline{\beta^{-1}} \le \underline{\alpha^{-1}}.$$

Proposition 7. We have

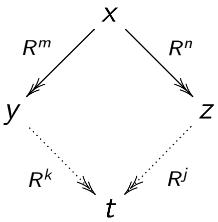
(1) $\overline{\omega} = \underline{\omega} = \mathrm{I}$

(2) $\underline{\iota} = \iota \leq \mathrm{I} \leq \overline{\iota}$

Characterization by L-fuzzy approximation operators

An L-fuzzy relation R is called *confluent* if

$$R^m(x,y) \odot R^n(x,z) \leq \bigvee_{t \in U} (R^k(y,t) \odot R^j(z,t)) \quad (m,n,k,j \in \mathbb{N}),$$



where R^n $(n \in \mathbb{N})$ is defined by

$$R^n = \begin{cases} \omega & (n = 0) \\ R \circ R^{n-1} & (n \geq 1). \end{cases}$$

Using the results

$$\overline{\mathrm{R}^{-1}} \dashv \underline{\mathrm{R}} \text{ and } \overline{\mathrm{R}} \dashv \underline{\mathrm{R}^{-1}},$$

we have the characterization theorem of the confluent by L-fuzzy operators $\overline{\mathrm{R}}$ and $\underline{\mathrm{R}}$.

Theorem 7. Let R be an L-fuzzy relation. Then we have

$$R \text{ is confluent } \iff \overline{\mathrm{R}}^{n} \underline{\mathrm{R}}^{j} \leq \underline{\mathrm{R}}^{m} \overline{\mathrm{R}}^{k}.$$

Proof Since

$$
\begin{aligned}
R \text{ is confluent} &\iff (R^{-1})^m \circ R^n \leq R^k \circ (R^{-1})^j \\
&\iff (\overline{\mathrm{R}^{-1}})^m \, \overline{\mathrm{R}}^n \leq \overline{\mathrm{R}}^k \, (\overline{\mathrm{R}^{-1}})^j,
\end{aligned}
$$

it is sufficient to show that

$$
\boxed{(\overline{\mathrm{R}^{-1}})^m \, \overline{\mathrm{R}}^n \leq \overline{\mathrm{R}}^k \, (\overline{\mathrm{R}^{-1}})^j \iff \overline{\mathrm{R}}^n \, \underline{\mathrm{R}}^j \leq \underline{\mathrm{R}}^m \, \overline{\mathrm{R}}^k.}
$$

$$
\begin{aligned}
&(\overline{\mathrm{R}^{-1}})^m \, \overline{\mathrm{R}}^n \leq \overline{\mathrm{R}}^k \, (\overline{\mathrm{R}^{-1}})^j \\
&\iff \overline{\mathrm{R}}^n \leq \underline{\mathrm{R}}^m \, \overline{\mathrm{R}}^k \, (\overline{\mathrm{R}^{-1}})^j \qquad (\because \ (\overline{\mathrm{R}^{-1}})^m \dashv \underline{\mathrm{R}}^m) \\
&\iff \overline{\mathrm{R}}^n \cdot \underline{\mathrm{R}}^j \leq \underline{\mathrm{R}}^m \, \overline{\mathrm{R}}^k \, (\overline{\mathrm{R}^{-1}})^j \cdot \underline{\mathrm{R}}^j \\
&\iff \overline{\mathrm{R}}^n \, \underline{\mathrm{R}}^j \leq \underline{\mathrm{R}}^m \, \overline{\mathrm{R}}^k \, (\overline{\mathrm{R}^{-1}})^j \cdot \underline{\mathrm{R}}^j \leq \underline{\mathrm{R}}^m \, \overline{\mathrm{R}}^k \cdot \mathrm{I} = \underline{\mathrm{R}}^m \, \overline{\mathrm{R}}^k
\end{aligned}
$$

Remark We note that, in the modal logic K, the confluent condition above corresponds to the formula

$$\Diamond^n \Box^j A \to \Box^m \Diamond^k A,$$

that is, the formula is characterized by the relation R satisfying the condition:

> *If $R^m(x, y)$ and $R^n(x, z)$, then there exists t such that $R^k(y, t)$ and $R^j(z, t)$.*

This expresses the correspondence between $\Box$ and $\underline{\mathrm{R}}$ (hence $\Diamond$ and $\overline{\mathrm{R}}$).

We define some types of L-fuzzy relations according to [3, 5]:

$$
\begin{aligned}
R \text{ is } \textit{reflexive} &\iff R(x,x) = 1;\\
R \text{ is } \textit{symmetric} &\iff R(x,y) = R(y,x);\\
R \text{ is } \textit{transitive} &\iff \bigvee_{z\in U}(R(x,z)\odot R(z,y)) \leq R(x,y);\\
R \text{ is } \textit{dense} &\iff R(x,y) \leq \bigvee_{z\in U}(R(x,z)\odot R(z,y));\\
R \text{ is } \textit{Euclidean} &\iff R(x,y)\odot R(x,z) \leq R(y,z);\\
R \text{ is } \textit{functional} &\iff R(x,y)\odot R(x,z) \leq \omega(y,z).
\end{aligned}
$$

Each property of L-fuzzy relations above can be represented only by R and R^{-1} as follows:

Proposition 8. Let R be an L-fuzzy relation. Then we have

$$\begin{aligned}
R \text{ is } \textit{reflexive} &\iff \omega \leq R; \\
R \text{ is } \textit{symmetric} &\iff R^{-1} \leq R; \\
R \text{ is } \textit{transitive} &\iff R \circ R \leq R; \\
R \text{ is } \textit{dense} &\iff R \leq R \circ R; \\
R \text{ is } \textit{Euclidean} &\iff R^{-1} \circ R \leq R; \\
R \text{ is } \textit{functional} &\iff R^{-1} \circ R \leq \omega.
\end{aligned}$$

From our theorem, we get the following results.

Corollary Let R be an L-fuzzy relation. Then we have

$$
\begin{aligned}
R \text{ is } \textit{reflexive} &\iff \mathrm{I} \le \overline{\mathrm{R}} \iff \underline{\mathrm{R}} \le \mathrm{I};\\
R \text{ is } \textit{symmetric} &\iff \mathrm{I} \le \underline{\mathrm{R}}\,\overline{\mathrm{R}} \iff \overline{\mathrm{R}}\,\underline{\mathrm{R}} \le \mathrm{I};\\
R \text{ is } \textit{transitive} &\iff \overline{\mathrm{R}}\,\overline{\mathrm{R}} \le \overline{\mathrm{R}} \iff \underline{\mathrm{R}} \le \underline{\mathrm{R}}\,\underline{\mathrm{R}};\\
R \text{ is } \textit{dense} &\iff \overline{\mathrm{R}} \le \overline{\mathrm{R}}\,\overline{\mathrm{R}} \iff \underline{\mathrm{R}}\,\underline{\mathrm{R}} \le \underline{\mathrm{R}};\\
R \text{ is } \textit{Euclidean} &\iff \overline{\mathrm{R}} \le \underline{\mathrm{R}}\,\overline{\mathrm{R}} \iff \overline{\mathrm{R}}\,\underline{\mathrm{R}} \le \underline{\mathrm{R}};\\
R \text{ is } \textit{functional} &\iff \overline{\mathrm{R}} \le \underline{\mathrm{R}}.
\end{aligned}
$$

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Thank you for your attention!