Diagonally Canonical Operations Do Not Capture Primitive-Positive Definability

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Outline

1. (Diagonally) Canonical Operations
2. The Question of Preservation
3. Reducts of the random graph and the generic triangle-free graph
4. Clones over $(\mathbb{N};=)$

Canonical Operations

Definition

Let $\mathfrak{A}$ be a relational structure. We denote by $\mathrm{Pol}(\mathfrak{A})$ *the set of all polymorphisms of* $\mathfrak{A}$, *i.e. homomorphisms from finite powers of* $\mathfrak{A}$ *to* $\mathfrak{A}$.

$$\mathrm{Pol}(\mathfrak{A}) = \bigcup_{n \geq 1} \mathrm{Hom}(\mathfrak{A}^n, \mathfrak{A}).$$

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- Canonical Polymorphisms are a restricted subset of all polymorphisms of a structure
- In many settings of ω-categorical infinite-domain CSPs, the clone of canonical polymorphisms witnesses tractability
- Examples include all CSPs of first-order reducts of the homogeneous universal poset ([Kompatscher, Pham '18]) and all CSPs of first-order reducts of homogeneous graphs ([BMPP '19]).

Canonical Operations

Definition

Let $\mathfrak{A}$ be an ω-categorical relational structure. An operation $f\colon \mathfrak{A}^k \to \mathfrak{A}$ is called **canonical over $\mathfrak{A}$** *if for all $n \in \mathbb{N}$ and $a_1, \ldots, a_k, b_1, \ldots, b_k \in A^n$, if*

$$\operatorname{typ}^{\mathfrak{A}}(a_i) = \operatorname{typ}^{\mathfrak{A}}(b_i)$$

for all $i \in \{1, \ldots, k\}$, then

$$\operatorname{typ}^{\mathfrak{A}}(f(a_1, \ldots, a_k)) = \operatorname{typ}^{\mathfrak{A}}(f(b_1, \ldots, b_k)).$$

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Example

Consider operations over $(\mathbb{Q}; <)$. An example for a canonical operation over $(\mathbb{Q}; <)$ is lex, *a binary injection on $\mathbb{Q}$ such that $\mathrm{lex}(a, b) < \mathrm{lex}(a', b')$ if either $a < a'$, or $a = a'$ and $b < b'$.*

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The operation lex being a polymorphism is not enough to guarantee tractability of a template!
None of the polymorphisms witnessing tractability of templates over $(\mathbb{Q}; <)$ is canonical!

Diagonally Canonical Operations

Definition

Let $\mathfrak{A}$ *be an* ω*-categorical relational structure. An operation* $f\colon \mathfrak{A}^k \to \mathfrak{A}$ *is called* **diagonally canonical over** $\mathfrak{A}$ *if for all* $n \in \mathbb{N}$ *and* $a_1, \ldots, a_k, b_1, \ldots, b_k \in A^n$*, if*

$$\mathrm{typ}^{\mathfrak{A}^k}((a_1, \ldots, a_k)) = \mathrm{typ}^{\mathfrak{A}^k}((b_1, \ldots, b_k)),$$

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Example

Let's revisit $(\mathbb{Q}; <)$. The polymorphisms min, mi, mx *and* ll, *witnessing tractability, all are diagonally canonical*[a]*, as well as their duals.*

[a] The classically defined ll is not diag.-can., but there is an equivalent polymorphism generated by and generating ll which is diag.-can., and which also witnesses tractability.

Preservation of Relations and Primitive Positive Definability

Definition

A first-order τ-formula $\phi(x_1, \ldots, x_n)$ is **primitive positive** *if it is of the form*

$$\exists x_{n+1}, \ldots, x_m(\psi_1 \wedge \cdots \wedge \psi_k),$$

where $\psi_1, \ldots, \psi_k$ are atomic τ-formulas.
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Theorem (Bodirsky, Nešetřil '03)

Let $\mathfrak{A}$ be a countable ω-categorical structure. A relation R has a primitive positive definition in $\mathfrak{A}$ if and only if R is preserved by all polymorphisms of $\mathfrak{A}$; in symbols,

$$R \in \mathrm{Inv}(\mathrm{Pol}(\mathfrak{A})) \iff R \in \langle \mathfrak{A} \rangle_{pp}.$$

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No. Counterexamples include model-complete core reducts of:

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No. Counterexamples include model-complete core reducts of:

- the random graph.
- the universal triangle-free graph (as well as the other Henson graphs),
- any k-neoliberal structure with free amalgamation.

Additionally, there are uncountably many different clones over $(\mathbb{N}; =)$ (up to pp-interdefinability) which are not distinguishable by diagonally canonical polymorphisms.

Reducts of the random graph

Theorem

Let $\Gamma = (V; E)$ *be the random graph. There exists a first-order reduct* $\mathfrak{A}$ *of* Γ *and a relation* R *such that* $\mathfrak{A}$ *is a model-complete core and all diagonally canonical polymorphisms of* $\mathfrak{A}$ *preserve* R*, but* R *is not primitively positively definable in* $\mathfrak{A}$*.*

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Proof (Idea).

We show the existence of clones $\mathscr{C}$, $\mathscr{D}$ such that

$$\mathrm{Aut}(\Gamma) \subseteq \mathscr{D} \subsetneq \mathscr{C}$$

and where $\mathscr{D}$ is generated by all diagonally canonical operations of $\mathscr{C}$.

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Let $\Gamma = (V; E)$ be the random graph. There exists a first-order reduct $\mathfrak{A}$ of Γ and a relation R such that $\mathfrak{A}$ is a model-complete core and all diagonally canonical polymorphisms of $\mathfrak{A}$ preserve R, but R is not primitively positively definable in $\mathfrak{A}$.

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We then choose $\mathfrak{A}$ such that $\mathrm{Pol}(\mathfrak{A}) = \mathscr{C}$. We get a first-order reduct of Γ with

$$\mathrm{Inv}(\mathscr{D}) \supsetneq \mathrm{Inv}(\mathscr{C}) = \langle \mathfrak{A} \rangle_{pp}.$$

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Proof (Idea).

To construct $\mathscr{C}$:

- Define a graph $\mathfrak{N}$ on V^2:
- Fix four vertices $s_1, s_2, t_1, t_2 \in V$ with $E(s_1, t_1)$ and $N(s_2, t_2)$.

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- For $(a_1, a_2), (b_1, b_2) \in V^2$, define $\mathfrak{N} \models E((a_1, a_2), (b_1, b_2))$ iff one of the following holds:
  - $\Gamma \models E(a_1, b_1)$ and $\Gamma \models E(a_2, b_2)$
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$\mathscr{D} := \overline{\langle \{g \in \mathscr{C} \mid g \text{ diag.-can.}\} \rangle}$ is a proper subset of $\mathscr{C}$. □

Clones over $(\mathbb{N}; =)$

We investigate the clones over $(\mathbb{N}; =)$.

Theorem (paraphrased) (Bodirsky, Chen, Pinsker '10)

There are uncountably many different clones (up to pp-interdefinability) over $(\mathbb{N}; =)$ containing the set of injective operations and all contained within $\mathrm{Pol}(\{\neq\})$.

Clones over $(\mathbb{N}; =)$

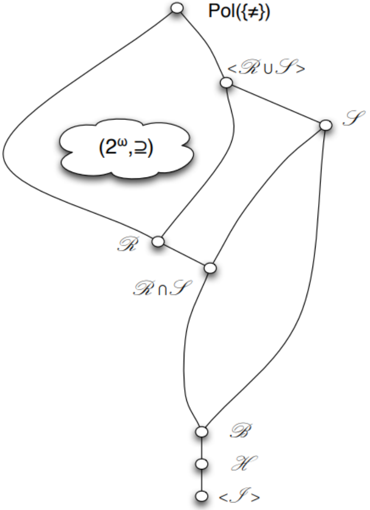

Figure: The clones over $(\mathbb{N}; =)$ whose unary part is injective ([BCP10]).

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Theorem

There are uncountably many different clones over $(\mathbb{N}; =)$*, all containing* $\mathscr{H}$*, which all have the same diagonally canonical operations.*

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Theorem

There are uncountably many different clones over $(\mathbb{N}; =)$, all containing $\mathscr{H}$, which all have the same diagonally canonical operations.

The proof follows from the following lemma:

Lemma

Let $\mathfrak{A}$ be a first-order reduct of $(\mathbb{N}; =)$. Let $f \in \mathrm{Pol}(\mathfrak{A})$ be diagonally canonical. Then f is either essentially unary or injective.

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  - For Ramsey structures, we know that having a Siggers polymorphism implies having a diagonally canonical Siggers polymorphism
- Understand what are the relations preserved by the diagonally canonical polymorphisms, but not pp-definable.
- In which situations do they determine tractability of the CSP?

Thank you for your attention

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