Semi-abelian varieties, commutators, and skew braces

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Outline

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Semi-abelian categories

Semi-abelian varieties

Homology of Skew Braces

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The notion of abelian category captures some crucial properties Ab and R-Mod have in common.

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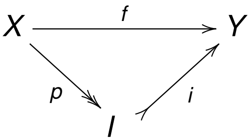
Definition

A category $\mathbb{C}$ is abelian if

- $\mathbb{C}$ has a zero-object 0
- $\mathbb{C}$ has binary products $A \times B$
- any arrow f in $\mathbb{C}$ has a factorization $f = i \circ p$



where p is a *normal epimorphism*, i is a *normal monomorphism*.

When $\mathbb{C}$ is pointed, an arrow $A \xrightarrow{p} P$ is a normal epimorphism if it is the cokernel of some arrow in $\mathbb{C}$: there is an i such that

$$\begin{array}{ccc} I & \xrightarrow{i} & A \\ \downarrow & & \downarrow p \\ 0 & \longrightarrow & P \end{array}$$

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is a pushout.

In the categories Ab of abelian groups and R-Mod of R-modules

- normal epimorphism = surjective homomorphism.

An arrow $K \xrightarrow{k} A$ is called a normal monomorphism if it is the kernel of some arrow in $\mathbb{C}$. This means that there is an $f\colon A \to B$ such that

$$\begin{array}{ccc} K & \overset{k}{\rightarrowtail} & A \\ \downarrow & & \downarrow{\scriptstyle f} \\ 0 & \longrightarrow & B \end{array}$$

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Groups

In the category **Grp** : normal monomorphism = normal subgroup inclusion

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Groups

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Abelian groups

In the category Ab : any monomorphism $k\colon K \to A$ is normal !

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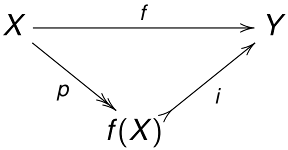
- Ab has a 0
- the product $A \times B$ exists for any A, B
- any homomorphism f in Ab has a factorization $f = i \circ p$



where p is a normal epimorphism and i a normal monomorphism.

Examples

Ab, R-Mod, Ab(Comp) are all abelian categories.

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What about the category Grp of groups ?

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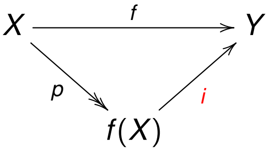
- the direct product $A \times B$ exists
- Problem : an arrow f in Grp does not have a factorization $f = i \circ p$



with p a *normal epimorphism* and i is a *normal* monomorphism.

Question : is there a list of simple axioms to conceptually understand some typical properties the categories Grp, Rng, Lie_K have in common ?

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Roughly speaking, the problem is to find the “fourth proportional” in

Ab : abelian category = Grp : ?

Aim : find an axiomatic context for

- Noether’s isomorphism theorems
- non-abelian homological algebra
- radical and torsion theories
- commutator theory

Historical remarks

- S. Mac Lane, Duality for groups, Bull. Amer. Math. Soc. (1950)

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- A. Grothendieck, Sur quelques points d'algèbre homologique, Tohoku Math. J. (1957)
- Several axiomatic proposals of "non-abelian contexts" :
  S. A. Amitsur (1954), A.G. Kurosh (1959),
  P. Higgins (1956), A. Frölich (1961), S.A. Huq (1968), M. Gerstenhaber (1970), O. Wyler (1971), G. Orzech (1972), etc.

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  b) regular epimorphisms are pullback stable,
  c) any equivalence relation is a kernel pair.
- $\mathbb{C}$ is (Bourn)-protomodular : the Split Short Five Lemma holds in $\mathbb{C}$

$$\begin{array}{ccccccc}
0 & \longrightarrow & K & \xrightarrow{k} & A & \underset{f}{\overset{s}{\leftrightarrows}} & B \\
 & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} \\
0 & \longrightarrow & K' & \xrightarrow[k']{} & A' & \underset{f'}{\overset{s'}{\leftrightarrows}} & B'
\end{array}$$

u, w isomorphisms $\Rightarrow$ v isomorphism.

Examples

Grp, Rng, Lie_K (more generally, any variety of Ω-groups in the sense of P. Higgins)

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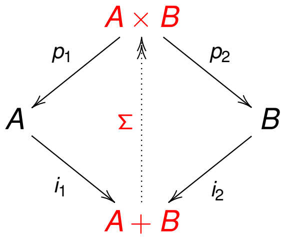
Any abelian category ! In particular : Ab, R-Mod, Ab(Comp), etc.

Remark

In semi-abelian categories the Noether isomorphism theorems hold. Moreover, one can develop non-abelian homological algebra and commutator theory.

In a semi-abelian category the canonical morphism Σ from $A + B$ to $A \times B$ is only a normal epimorphism (not an isomorphism !)



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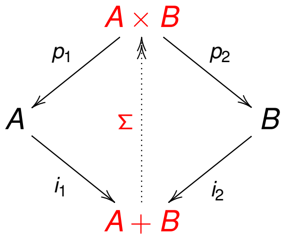

A semi-abelian category is not additive ! In particular : $A \times B \ncong A \oplus B$.

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Whereas

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the “non-additive” version of this “equation” is

semi-abelian = exact + 0 + *protomodular*.

One replaces the “additivity” with the validity of the “Split Short Five Lemma”.

Any semi-abelian category $\mathbb{C}$ contains the abelian category $\mathsf{Ab}(\mathbb{C})$ of its (internal) abelian groups as a Birkhoff subcategory

$$\mathsf{Ab}(\mathbb{C}) \underset{U}{\overset{\mathsf{ab}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbb{C}$$

where the universal quotient is

$$A \longrightarrow \mathsf{ab}(\mathsf{A}) = \frac{A}{[A,A]} \ ,$$

with $[A, A]$ the largest (categorical) commutator.

Outline

The varieties that are abelian are the categories of R-modules over a unitary ring.
Among varieties of universal algebras, which are the ones that are semi-abelian ?

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Theorem (D. Bourn and G. Janelidze, Theory Appl. Categories, 2003)

A variety $\mathbb{V}$ of universal algebras is semi-abelian if and only if its theory has a unique constant 0, an n-ary term β, and $n+1$ binary terms $\alpha_i(x, y)$ such that

$$\alpha_i(x, x) = 0$$

for any $i \in \{1, \cdots, n\}$, and

$$\beta(\alpha_1(x, y), \cdots, \alpha_n(x, y), y) = x.$$

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Example

In the variety Grp of groups $\beta(x,y) = x + y$ and $\alpha_1(x,y) = x - y$.

Remark

The varieties having the terms satisfying these identities had already been considered by A. Ursini (*Algebra Universalis*, 1994), under the name of classically ideal determined varieties.

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Besides the varieties of groups, rings, associative algebras, Lie algebras, etc. there are some classically ideal determined varieties of importance in logic, such as the variety of Heyting semilattices.

Outline

Homology of Skew Braces

L. Guarnieri and L. Vendramin (Math. Comp., 2017) introduced the structure of skew (left) brace, that produce *bijective* solutions of the *Yang-Baxter equation*, i.e. pairs (X, r) such that X is a set, and $r\colon X \times X \to X \times X$ is bijective and satisfies

$$(r \times Id_X) \cdot (Id_X \times r) \cdot (r \times Id_X) = (Id_X \times r) \cdot (r \times Id_X) \cdot (Id_X \times r).$$

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Definition

A skew brace is a set A with two group structures, $(A, +)$ and $(A, \circ)$ such that

$$a \circ (b + c) = a \circ b - a + a \circ c, \qquad \forall a, b, c \in A.$$

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The solutions of the Y.-B. equation are given by $r\colon A \times A \to A \times A$ defined by

$$r(a, b) = (-a + a \circ b, (-a + a \circ b)' \circ a \circ b),$$

where $(-a + a \circ b)'$ is the inverse of $-a + a \circ b$ for the $\circ$ operation.

Remark

SKB is a semi-abelian variety of algebras : $\beta(x, y) = x + y$ and $\alpha_1(x, y) = x - y$.

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SKB contains the variety Grp of groups as a subvariety : any group $(G, +)$ can be seen as a skew brace $(G, +, +)$. Up to this identification of $(G, +)$ with $(G, +, +)$, one has the adjunction

$$\mathsf{Grp} \underset{U}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathsf{SKB}$$

SKB contains the variety of RadRng of radical rings as a subvariety :

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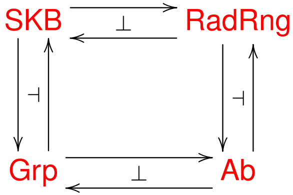
Note that the "$\circ$" operation of a radical ring $(A,+,\cdot)$ is defined by

$$a\circ b = a + a\cdot b + b.$$

We have the following subvarieties :



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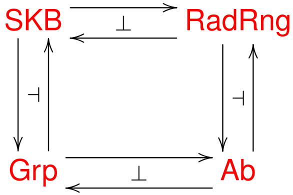

One can then study the homology of skew braces with “coefficients” in each of the subvarieties RadRng, Grp and Ab.

Definition

A subset R of a skew brace $(A, +, \circ)$ is an ideal if $R \triangleleft (A, +)$, $R \triangleleft (A, \circ)$, and

$$a + R = a \circ R \qquad \forall a \in A.$$

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Given any skew brace $(A, +, \circ)$ the universal group associated with it is the quotient

$$A \longrightarrow \frac{A}{A * A}$$

where

$$A * A = \langle a * b \mid a \in A, b \in A \rangle_A$$

and

$$a * b = -a + a \circ b - b.$$

Given any ideal R of $(A, +, \circ)$, we define

$$[R, A]_{\text{Grp}} = \langle \{a * b, b * a, c + b * a - c \mid a \in R, b \in A, c \in A\} \rangle_A.$$

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Given any free presentation

$$0 \longrightarrow R \longrightarrow F \longrightarrow A \longrightarrow 0$$

of a skew brace $(A, +, \circ)$, the expression

$$\frac{R \cap (F * F)}{[R, F]_{\mathsf{Grp}}}$$

turns out to be independent of the chosen free presentation.

Given two free presentations

$$0 \longrightarrow R \longrightarrow F \longrightarrow A \longrightarrow 0$$

and

$$0 \longrightarrow R' \longrightarrow F' \longrightarrow A \longrightarrow 0$$

of the skew brace $(A, +, \circ)$, one always has the isomorphism

$$\frac{R \cap (F * F)}{[R, F]_{\mathrm{Grp}}} \cong \frac{R' \cap (F' * F')}{[R', F']_{\mathrm{Grp}}}.$$

For any skew brace $(A, +, \circ)$ with free presentation

$$0 \longrightarrow R \longrightarrow F \longrightarrow A \longrightarrow 0 \tag{1}$$

one then defines

$$\mathrm{H}_1(A) := \frac{A}{A * A}$$

and

$$\mathrm{H}_2(A) := \frac{R \cap (F * F)}{[R, F]_{\mathrm{Grp}}}.$$

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H_1 and H_2 are the first and second homology functors.

Theorem (M. G., T. Letourmy, L. Vendramin, J. Pure Appl. Algebra 2025)
Any short exact sequence

$$0 \longrightarrow K \longrightarrow A \xrightarrow{f} B \longrightarrow 0 \qquad (2)$$

of skew braces - with $A \cong \frac{F}{R}$ and $B \cong \frac{G}{S}$ - induces a 5-term exact sequence

$$\frac{R \cap (F * F)}{[R,F]_{\mathrm{Grp}}} \xrightarrow{H_2(f)} \frac{S \cap (G * G)}{[S,G]_{\mathrm{Grp}}} \longrightarrow \frac{K}{[K,A]_{\mathrm{Grp}}} \longrightarrow \frac{A}{A * A} \xrightarrow{H_1(f)} \frac{B}{B * B} \longrightarrow 0$$

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Remark
This exact sequence is similar to the Stallings-Stammbach exact sequence in group theory, where $[R, F]_{\mathrm{Grp}}$ is replaced by $[R, F]$, and $G * G$ by $[G, G]$.

The subvariety of radical rings

Radical rings form a subvariety RadRng of the variety SKB of skew braces.
Given elements a, b, c in a skew brace $(A, +, \circ)$ we define their "right distributor" :

$$[a, b, c] = (a + b) \circ c - b \circ c + c - a \circ c.$$

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We call radicalator $[A, A]_{\mathrm{RadRng}}$ of A the following additive subgroup :

$$[A, A]_{\mathrm{RadRng}} = \langle \{ [a, b, c], [a, b]_{+} \mid a, b, c \in A \} \rangle_A.$$

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Lemma

$[A, A]_{\mathrm{RadRng}}$ is an ideal of $(A, +, \circ)$, and the quotient

$$A \xrightarrow{\eta_A} \frac{A}{[A,A]_{\mathrm{RadRng}}}$$

gives the universal radical ring associated with $(A, +, \circ)$.

This means that, given any homomorphism $f\colon A \to B$ from a skew brace $(A, +, \circ)$ to a radical ring $(B, +, \circ)$, there is a unique homomorphism $\frac{A}{[A,A]_{\mathsf{RadRng}}} \overset{\bar{f}}{\dashrightarrow} B$ making the following triangle commute :

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \frac{A}{[A,A]_{\mathsf{RadRng}}} \\ & {\scriptstyle \forall f} \searrow & \Big\downarrow {\scriptstyle \exists ! \bar{f}} \\ & & B \end{array}$$

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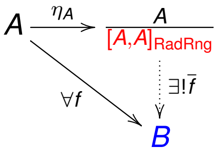

Given an ideal R of a skew brace $(A, +, \circ)$, one defines the commutator $[R, A]_{\mathrm{RadRng}}$ as follows :

$$[R, A]_{\mathrm{RadRng}} = \langle \{[a, b, c], [c, a, b], [b, c, a], [a, b]_+ \mid a \in R, b, c \in A\} \rangle_A.$$

The subvariety of abelian groups

We consider the subvariety Ab of abelian groups : $\mathrm{Ab} \underset{U}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{SKB}$, where, given any skew brace $(A, +, \circ)$, the relevant commutator is

$$[A, A]_{\mathrm{Ab}} = \langle \{[a, b]_+, a * b \mid a \in A, b \in A\} \rangle_A.$$

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$$[A, A]_{\mathsf{Ab}} = \langle \{ [a, b]_+, a * b \mid a \in A, b \in A \} \rangle_A.$$

The quotient of A by this ideal yields the universal abelian group associated with the skew brace $(A, +, \circ)$: for any homomorphism $A \xrightarrow{f} B$ to an abelian group $(B, +, +)$ there is a unique homomorphism $\overline{f}$ s. t. the following triangle commutes :

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \dfrac{A}{[A,A]_{\mathsf{Ab}}} \\ & \underset{\forall f}{\searrow} & \Big\downarrow {\scriptstyle \exists ! \overline{f}} \\ & & B \end{array}$$

For any ideal I of a skew brace $(A, +, \circ)$ one sets

$$[I, A]_{\mathrm{Ab}} = \langle \{ [a, b]_+, a * b, [a, b]_\circ \mid a \in I, b \in A \} \rangle_A.$$

The subgroup $[I, A]_{\mathrm{Ab}}$ is actually an ideal of $(A, +, \circ)$.

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The subgroup $[I, A]_{\mathrm{Ab}}$ is actually an ideal of $(A, +, \circ)$.

Lemma

$[I, A]_{\mathrm{Ab}} = [I, A]_{\mathrm{Huq}}$, where $[I, A]_{\mathrm{Huq}}$ is the Huq commutator of I and A.

For any ideal I of a skew brace $(A,+,\circ)$ one sets

$$[I,A]_{\mathrm{Ab}} = \langle\{[a,b]_+,\, a*b,\, [a,b]_\circ \mid a\in I, b\in A\}\rangle_A.$$

The subgroup $[I,A]_{\mathrm{Ab}}$ is actually an ideal of $(A,+,\circ)$.

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The Huq commutator $[I,A]_{\mathrm{Huq}}$ is the smallest ideal J of $(A,+,\circ)$ such that the composite homomorphism

$$I\times A \xrightarrow{\;c\;} A \xrightarrow{\;q_J\;} A/J$$

is a homomorphism of skew braces, where $c(i,a) = i+a$ (for any $i\in I$, $a\in A$), and $q_J\colon A\to A/J$ is the canonical quotient of A by J.

Theorem (M. G., T. Letourmy, L. Vendramin, J. Pure Appl. Algebra 2025)

Any short exact sequence

$$0 \longrightarrow K \longrightarrow A \xrightarrow{f} B \longrightarrow 0$$

of skew braces - with free presentations $A \cong \frac{F}{R}$ and $B \cong \frac{G}{S}$ - induces a 5-term exact sequence

$$\frac{R \cap ([F,F]_{\mathrm{Ab}})}{[R,F]_{\mathrm{Ab}}} \xrightarrow{\mathrm{H}_2(f)} \frac{S \cap [G,G]_{\mathrm{Ab}}}{[S,G]_{\mathrm{Ab}}} \longrightarrow \frac{K}{[K,A]_{\mathrm{Ab}}} \longrightarrow \frac{A}{[A,A]_{\mathrm{Ab}}} \xrightarrow{\mathrm{H}_1(f)} \frac{B}{[B,B]_{\mathrm{Ab}}} \longrightarrow 0$$

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Finally, one can define the lower central series $A^0 = A$, $A^1 = [A, A]_{\mathrm{Ab}}$, ... ,

$$A^{n+1} = [A, A^n]_{\mathrm{Ab}}, \qquad \forall n \geq 1.$$

By applying similar methods as the ones above to the short exact sequence

$$0 \longrightarrow A^n \longrightarrow A \xrightarrow{f} \frac{A}{A^n} \longrightarrow 0$$

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Theorem (M. G., T. Letourmy, L. Vendramin, J. Pure Applied Algebra 2025)

Let $f\colon A \to B$ be a morphism of skew braces. If

- the homomorphism $\mathrm{H}_1(f)\colon \mathrm{H}_1(A) \xrightarrow{\cong} \mathrm{H}_1(B)$ is an isomorphism,
- the homomorphism $\mathrm{H}_2(f)\colon \mathrm{H}_2(A) \longrightarrow \mathrm{H}_2(B)$ is surjective,

then, for any $n \geq 1$, the induced homomorphism

$$\frac{A}{A^n} \xrightarrow{\cong} \frac{B}{B^n}$$

is an isomorphism.

Final remarks

- It would be interesting to establish Hopf formulae for the higher-order homology of skew braces by using the Categorical Galois theory developed by G. Janelidze (J. Algebra, 1990).

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- There are many other applications of these methods allowing one to use commutators to compute the homology of other algebraic and topological structures, such as crossed modules, compact groups, internal n-groupoids, etc. (T. Everaert, M.G. and T. Van der Linden, Adv. Math., 2008, and N. Egner, J. Algebra, 2026)

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- These results have a natural counterpart in the categories of cocommutative Hopf algebras (M.G. and A. Sciandra, Annali Matem. Pura Applicata, 2026).

References

- S. Mac Lane, *Duality for groups*, Bull. Amer. Math. Soc. (1950)
- G. Janelidze, *Pure Galois theory in Categories*, J. Algebra (1990)
- A. Ursini, *On subtractive varieties, I*, Algebra Univers. (1994)
- G. Janelidze, L. Márki and W. Tholen, *Semi-abelian categories*, J. Pure Appl. Algebra (2002)
- D. Bourn and G. Janelidze, *Characterization of protomodular varieties of universal algebras*, Theory Appl. Categories (2003)
- T. Everaert, M. Gran and T. Van der Linden, *Higher Hopf Formulae for Homology via Galois Theory*, Adv. Math. (2008)
- L. Guarnieri, L. Vendramin, *Skew braces and the Yang-Baxter equation*, Math. Comp. (2017)
- M. Gran, T. Letourmy and L. Vendramin, *Hopf formulae for homology of skew braces*, J. Pure Appl. Algebra (2025)
- N. Egner, *Galois theory and homology in quasi-abelian functor categories,* J. Algebra (2025)
- M. Gran and A. Sciandra, *Hopf formulae for cocommutative Hopf algebras*, Annali Matem. Pura Applicata, (2026), accepted for publication.