# DECIDABILITY OF INTERPRETABILITY

ROMAN FELLER

TU WIEN

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Equivalence rel. of $\omega$-cat. structures ( $\mathbb{A}$, $\mathbb{B}$ living on $\omega$)

its first-order theory has unique countable model

# Equivalence rel. of $\omega$-cat. structures ($\mathbb{A}$, $\mathbb{B}$ living on $\omega$)

Engeler, Ryll-Nardzewski, Svenonius

first-order

coarser

inter-def. $\mathrm{Aut}\,\mathbb{A} = \mathrm{Aut}\,\mathbb{B}$

bi-def. $\mathrm{Aut}\,\mathbb{A} \cong_{conj} \mathrm{Aut}\,\mathbb{B}$

$:\Leftrightarrow \exists \phi \in \mathrm{Sym}(\omega)$
$\mathrm{Aut}\,\mathbb{B} = \phi(\mathrm{Aut}\,\mathbb{A})\phi^{-1}$

bi-int. $\mathrm{Aut}\,\mathbb{A} \cong_{top} \mathrm{Aut}\,\mathbb{B}$

Coquand (in Ahlbrandt & Ziegler)

$\mathrm{Aut}\,\mathbb{A} \cong_{alg} \mathrm{Aut}\,\mathbb{B}$

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RECONSTRUCTION

"Guide to top. reconstruction" (Marimon + Pinsker 2025)

# Pol and the Algebraic Approach to CSPs

CSP$\mathbb{A}$ computational problem:

$$\boxed{\begin{array}{l} \text{INPUT: finite rel. structure } \mathbb{I} \\[1em] \text{QUESTION: } \mathbb{I} \xrightarrow{\text{hom.}} \mathbb{A}\ ? \end{array}}$$

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$\mathbb{A}$ $\omega$-cat:

- $\mathrm{Pol}\,\mathbb{A} = \mathrm{Pol}\,\mathbb{B} \Rightarrow$

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Algebraic approach:

"Study polymorphism clones up to top. isom. to understand complexity of CSPs"

# Measuring complexity of equivalence relations

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## Borel reductions

- $X, Y$ Borel spaces (e.g. $2^\omega \cong_{Borel} \mathbb{R}$)
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- $E_X$ is smooth if it Borel-reduces to $(2^\omega, =)$.

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## Computable reductions

- $X, Y \subseteq \omega$,
- $f$ computable function
- $E_X$ is decidable if it comp. reduces to $(\omega, =)$.

| | | | |
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- assignment $I : \mathbb{A} \mapsto \mathrm{Th}(\mathcal{O}_{\mathrm{Aut}\,\mathbb{A}})$ is Borel
  - $\mathcal{O}_{\mathrm{Aut}\,\mathbb{A}} = (\omega ; (\sim_n)_{n \in \omega})$   $x \sim_n y :\Leftrightarrow x, y \in \omega^n + \exists \alpha \in \mathrm{Aut}\,\mathbb{A}$ $\alpha(x) = y$.
  - $\mathcal{O}_{\mathrm{Aut}\,\mathbb{A}} \cong \mathcal{O}_{\mathrm{Aut}\,\mathbb{B}} \Leftrightarrow \mathrm{Aut}\,\mathbb{A} \cong_{conj} \mathrm{Aut}\,\mathbb{B} + \mathcal{O}_{(\ldots)}$ $\omega$-cat.

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(F. + Pinsker 2025/ongoing) Smooth if restricted to NO ALGEBRAICITY*

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# Computational complexity?

# Decidability of Definability

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finite

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Example: Random graph, $(\mathbb{Q}, <)$, Henson's generic $K_n$-free graph, ...

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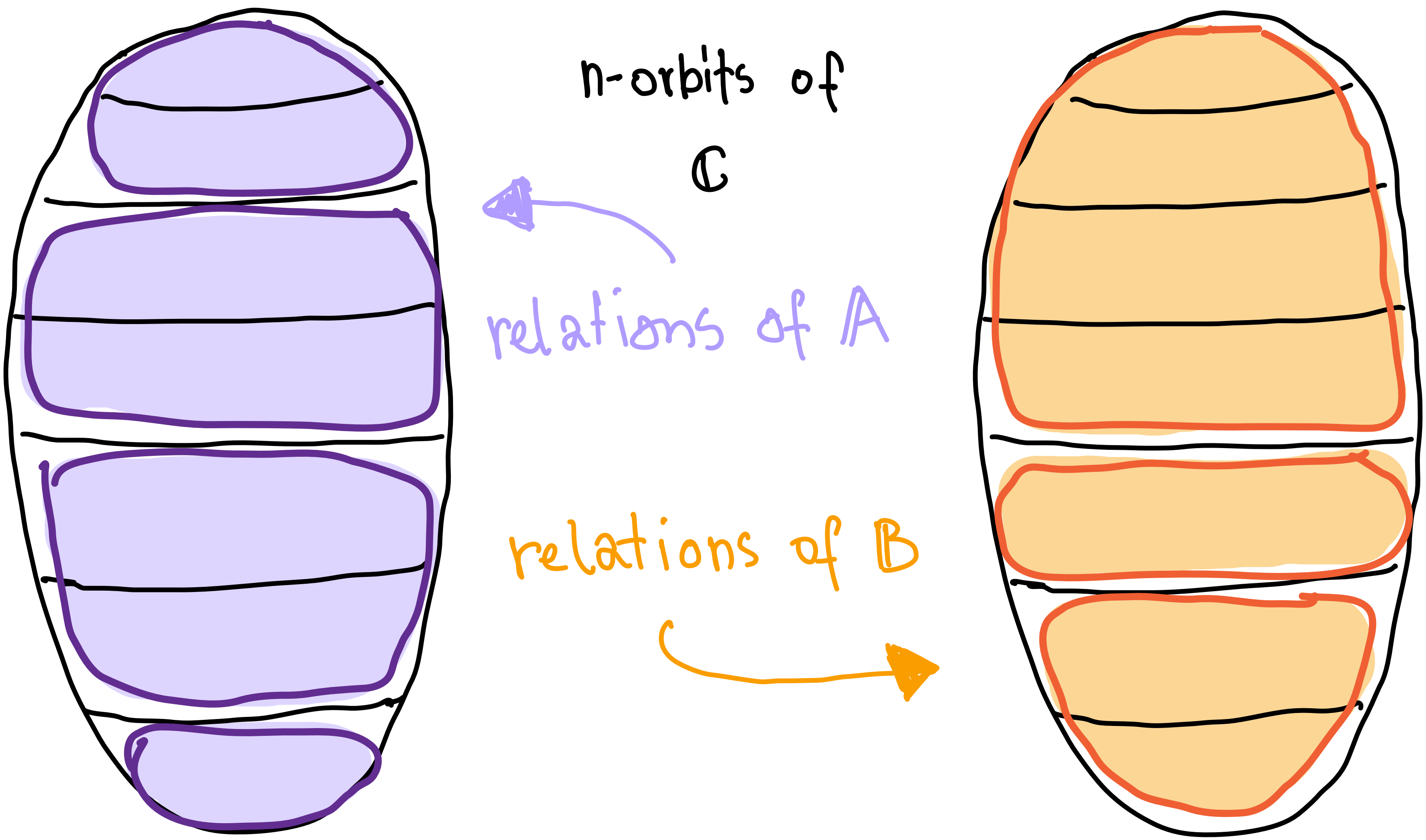
- $\mathbb{C} = \mathrm{Flim}(\mathrm{Forb}(\underset{\text{finite}}{\mathcal{F}}))$ (can list all n-orbits of $\mathbb{C}$)
- fin. language fo-reducts $\mathbb{A}, \mathbb{B}$ of $\mathbb{C}$ (relations unions of $\mathbb{C}$-orb.)



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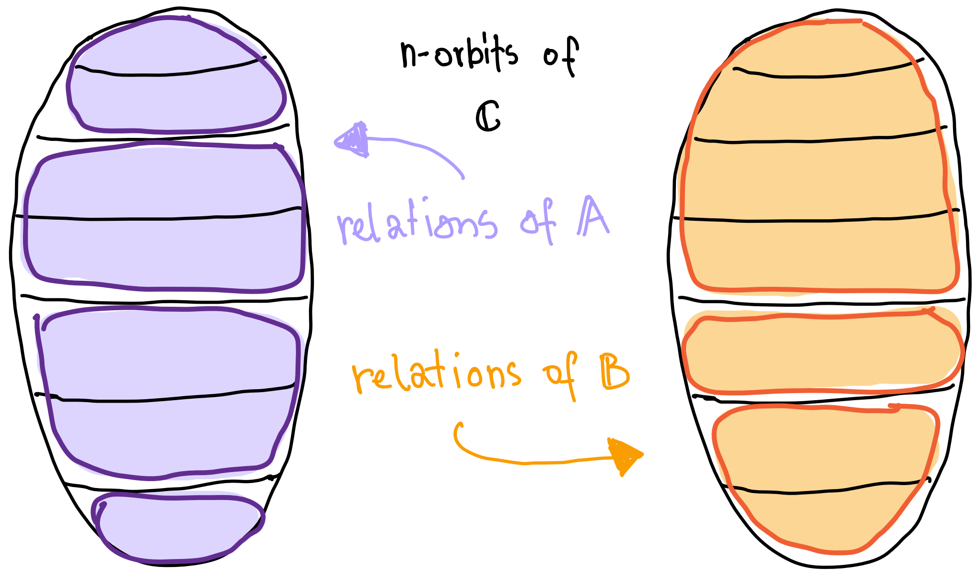


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Too difficult!

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- fin. language fo-reducts $\mathbb{A}, \mathbb{B}$ of $\mathbb{C}$ (relations unions of $\mathbb{C}$-orb.)

Question:
- $\mathrm{Aut}\,\mathbb{A} = \mathrm{Aut}\,\mathbb{B}$ ?
- $\mathrm{End}\,\mathbb{A} = \mathrm{End}\,\mathbb{B}$ ?
- $\mathrm{Pol}\,\mathbb{A} = \mathrm{Pol}\,\mathbb{B}$ ?

Too difficult!

Add Ramsey

# Decidability of Definability

- $\mathbb{C} = \mathrm{Flim}(\mathrm{Forb}(\mathcal{F}))$    Ramsey structure

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(Bodirsky + Pinsker + Tsankov 2011)

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**pf sketch.** $\mathrm{End}\,\mathbb{A} \neq \mathrm{End}\,\mathbb{B} \iff$ wlog $\exists f \in \mathrm{End}\,\mathbb{A} \setminus \mathrm{End}\,\mathbb{B}$

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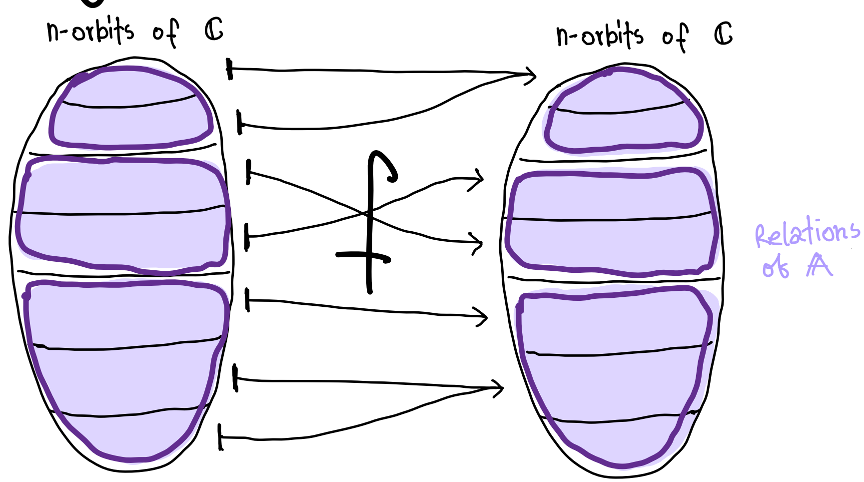

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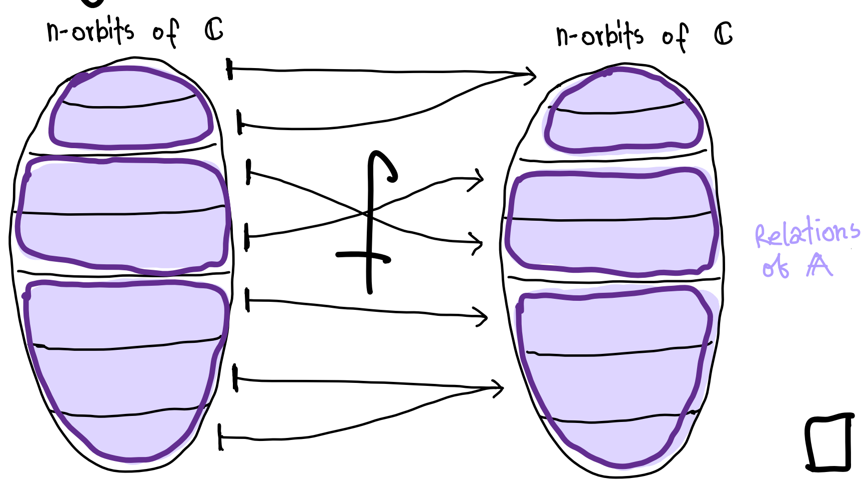


$\square$

# Decidability of Interpretability

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$\mathbb{A}$ model-complete core $:\Leftrightarrow$ $\overline{\mathrm{Aut}\,\mathbb{A}} = \mathrm{End}\,\mathbb{A}$

$\boxed{\text{(Bodirsky 2005) } \mathbb{A} \text{ } \omega\text{-cat. } \exists\, \mathbb{A}^{core} \text{ m-c core hom. equiv. to } \mathbb{A}. \; \mathbb{A}^{core} \text{ unique up to isom. + } \omega\text{-cat.} \; (\mathrm{CSP}\,\mathbb{A} = \mathrm{CSP}\,\mathbb{A}^{core})}$

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Example: the model-complete core of

- $(\mathbb{Q}, <)$ is $(\mathbb{Q}, <)$
- the random graph is $K_\omega$
- $(\mathbb{Q}, \leq)$ is a point with equality
- $K_{\omega,\omega}$ is $K_2$

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(F. + Pinsker 2025) $\mathbb{A}$, $\mathbb{B}$ fo-reducts of $\mathrm{Flim}(\mathrm{Forb}\,\mathcal{F})$, $\mathrm{Flim}(\mathrm{Forb}\,\mathcal{G})$ Ramsey, without algebraicity.

Can decide:

- $\mathrm{End}\,\mathbb{A}^{core} \cong_{top} \mathrm{End}\,\mathbb{B}^{core}$
- $\mathrm{Pol}\,\mathbb{A}^{core} \cong_{top} \mathrm{Pol}\,\mathbb{B}^{core}$ *

* if $\mathbb{A}$, $\mathbb{B}$ additionally transitive; use Behrisch, Vargas-García (2019)

# Thank you!

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