Tame ω-categoricity and CSPs

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Constraint satisfaction problems

Definition

$\mathfrak{B}$: a structure with a finite relational signature.
CSP($\mathfrak{B}$) is the following decision problem.

- INPUT: a finite structure $\mathfrak{A}$ (with the same signature as $\mathfrak{B}$)
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Alternative formulation

- INPUT: A primitive positive (pp) sentence φ over $\mathfrak{B}$ ($\varphi \equiv \exists\exists \ldots \exists(\bigwedge(\text{atomic}))$)
- QUESTION: Is φ true in $\mathfrak{B}$?

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$$\{(a_1^1, \ldots, a_d^1, \ldots, a_1^k, \ldots, a_d^k) : (I(a_1), \ldots, I(a_k)) \in R\}$$

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Fact

If $\mathfrak{A}$ *pp-interprets* $\mathfrak{B}$ *then* CSP($\mathfrak{B}$) LOGSPACE *reduces to* CSP($\mathfrak{A}$).

pp-constructions

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$\mathfrak{A}$ and $\mathfrak{B}$ are homomorphically equivalent iff there are homomorphisms $\mathfrak{A} \to \mathfrak{B}$ and $\mathfrak{B} \to \mathfrak{A}$.

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Theorem (Siggers '10)

If $\mathfrak{B}$ is finite then $\mathfrak{B}$ not omniexpressive iff Pol($\mathfrak{B}$) *contains a Siggers operation:* $s(x, y, x, z, y, z) = s(y, x, z, x, z, y)$.

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Corollary

If $\mathfrak{B}$ is finite then CSP($\mathfrak{B}$) *is in* **P** *or it is* **NP***-complete.*

ω-categoricity

Definition (the useful one)

$\mathfrak{B}$ is ω-categorical if $\mathrm{Aut}(\mathfrak{B})$ has finitely many n-orbits for all $n \in \omega$.

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Theorem (Bodirsky, Nešetřil '03)

If $\mathfrak{B}$ is ω-categorical, then the complexity of $\mathrm{CSP}(\mathfrak{B})$ *is uniquely determined by* $\mathrm{Pol}(\mathfrak{B})$.

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Theorem (Bodirsky '05)

Every ω-categorical structure is homomorphically equivalent to a model-complete core.
This is unique up to isomorphism, and again ω-categorical.

Infinite-domain CSP dichotomy

Theorem (Barto, Pinsker '20)

$\mathfrak{B}$ is an ω-categorical model-complete core which is not omniexpressive. Then $\mathrm{Pol}(\mathfrak{B})$ contains a pseudo-Siggers operation:
$(\alpha \circ s)(x, y, x, z, y, z) = (\beta \circ s)(y, x, z, x, z, y) : \alpha, \beta \in \overline{\mathrm{Aut}(\mathfrak{A})}$.

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Conjecture (Bodirsky, Pinsker)

If $\mathfrak{B}$ is FOROFBHS* and $\mathfrak{B}$ is not omniexpressive then CSP($\mathfrak{B}$) is in **P**.

*first-order reduct of a finitely bounded homogeneous structure

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Known CSP dichotomies

Solved for

- reducts of $(\mathbb{N}; =)$ (Bodirsky, Kára '08)
- reducts of $(\mathbb{Q}; <)$ (Bodirsky, Kára '09)
- reducts of the homogeneous binary branching C-structure (Bodirsky, Jonsson, Pham '16)
- reducts of homogeneous graphs (Bodirsky, Martin, Pinsker, Pongrácz '19)
- reducts of the random poset (Kompatscher, Pham '18)
- reducts of unary ω-categorical structures (Bodirsky, Mottet '18)
- MMSNPs (Bodirsky, Madelaine, Mottet '18)
- reducts of the random tournament (Mottet, Pinsker '21)
- first-order expansions of the homogeneous RCC5 structure (Bodirsky, B. '21)
- hereditarily cellular structures (B. '22)
- first-order expansions of powers of $(\mathbb{Q}; <)$ (Bodirsky, Jonsson, Martin, Mottet, Semanišinová '22)
- reducts of random uniform hypergraphs (Mottet, Nagy, Pinsker '23)
- reducts of Johnson graphs (Bodirsky, B. '25)

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- Solve the dichotomy for “building blocks” (primitive structures).
- Put the pieces together. (???)

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3. From second-order logic: MMSNP, GMSNP
4. First-order interpretability in certain structures; mostly $(\mathbb{N}; =)$ or $(\mathbb{Q}; <)$.

Interpretation of structures

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Notation

$\mathrm{I}_{fo}(\mathfrak{A})$: structures first-order interpretable in $\mathfrak{A}$.

Problem with interpretations

Facts

- ☹ $\mathrm{I}_{fo}((\mathbb{N};=))$ is not closed under taking model-complete core. (Bodirsky, B., Marimon '25)

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Lachlan's class

Definition/Theorem (Lachlan '87)

$\mathfrak{B} \in \mathcal{D}$ (Lachlan's class) iff

- $\mathfrak{B} \in \mathrm{I}_{fo}(\mathbb{Q}; <)$, and
- no preorder with infinite chains is definable on tuples in $\mathfrak{B}$ (NSOP).

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$\mathrm{I}_{fo}(\mathbb{N}; =) \subset \mathcal{D} \subset \mathrm{I}_{fo}(\mathbb{Q}; <)$.

Theorem (Cherlin, Lachlan, Harrington '85+Bodirsky, B., Marimon '26+)

TFAE.

1. $\mathfrak{B} \in \mathcal{D}$ *and primitive.*

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Primitive structures

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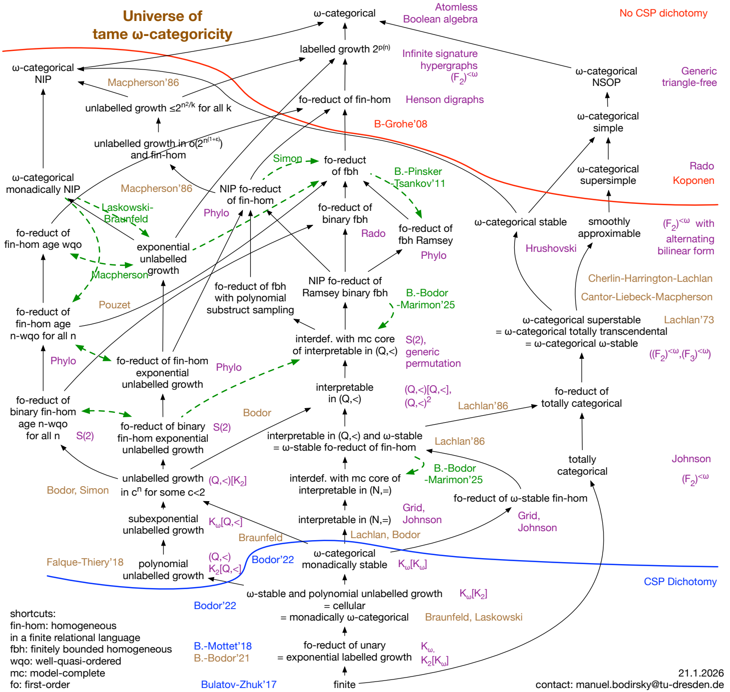
Theorem (Bodirsky, B., Marimon '26+)

Every model-complete core as in item 3 is omniexpressive unless $n = k = 1$ *(independent of its polymorphisms).*


Universe of tame ω-categoricity
ω-categorical
Atomless Boolean algebra
No CSP dichotomy
labelled growth $2^{p(n)}$
Infinite signature hypergraphs $(F_2)^{<\omega}$
ω-categorical NIP
Macpherson'86
unlabelled growth $\leq 2^{n^2/k}$ for all k
fo-reduct of fin-hom
Henson digraphs
ω-categorical NSOP
Generic triangle-free
ω-categorical simple
B-Grohe'08
unlabelled growth in $o(2^{n^{(1+\epsilon)}})$ and fin-hom
Simon
fo-reduct of fbh
B.-Pinsker -Tsankov'11
ω-categorical supersimple
Rado
Koponen
ω-categorical monadically NIP
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NIP fo-reduct of fin-hom
Laskowski-Braunfeld
Phylo
fo-reduct of binary fbh
Rado
fo-reduct of fbh Ramsey
Phylo
ω-categorical stable
smoothly approximable
$(F_2)^{<\omega}$ with alternating bilinear form
fo-reduct of fin-hom age wqo
exponential unlabelled growth
Macpherson
Hrushovski
Cherlin-Harrington-Lachlan
Cantor-Liebeck-Macpherson
fo-reduct of fbh with polynomial substruct sampling
NIP fo-reduct of Ramsey binary fbh
B.-Bodor -Marimon'25
Pouzet
fo-reduct of fin-hom age n-wqo for all n
ω-categorical superstable = ω-categorical totally transcendental = ω-categorical ω-stable
Lachlan'73
$((F_2)^{<\omega},(F_3)^{<\omega})$
interdef. with mc core of interpretable in (Q,<)
S(2), generic permutation
Phylo
fo-reduct of fin-hom exponential unlabelled growth
Phylo
interpretable in (Q,<)
(Q,<)[Q,<], $(Q,<)^2$
Lachlan'86
fo-reduct of totally categorical
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S(2)
Bodor
interpretable in (Q,<) and ω-stable = ω-stable fo-reduct of fin-hom
Lachlan'86
totally categorical
Johnson
unlabelled growth in c^n for some c<2
$(Q,<)[K_2]$
interdef. with mc core of interpretable in (N,=)
B.-Bodor -Marimon'25
$(F_2)^{<\omega}$
Bodor, Simon
fo-reduct of ω-stable fin-hom
subexponential unlabelled growth
$K_\omega[Q,<]$
interpretable in (N,=)
Grid, Johnson
Grid, Johnson
Braunfeld
Lachlan, Bodor
Falque-Thiery'18
polynomial unlabelled growth
(Q,<)
$K_2[Q,<]$
Bodor'22
ω-categorical monadically stable
$K_\omega[K_\omega]$
CSP Dichotomy
ω-stable and polynomial unlabelled growth = cellular = monadically ω-categorical
$K_\omega[K_2]$
Bodor'22
Braunfeld, Laskowski
shortcuts:
fin-hom: homogeneous in a finite relational language
fbh: finitely bounded homogeneous
wqo: well-quasi-ordered
mc: model-complete
fo: first-order
B.-Mottet'18
B.-Bodor'21
fo-reduct of unary = exponential labelled growth
K_ω, $K_2[K_\omega]$
Bulatov-Zhuk'17
finite
21.1.2026
contact: manuel.bodirsky@tu-dresden.de

Model theoretical tameness: the picture

Link to the picture:

https://wwwpub.zih.tu-dresden.de/~bodirsky/Tame-omega-categoricity.pdf