Embedding monoids in residuated lattices

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This problem can really be seen in triplicate:

Problem. Let V be a variety of residuated lattices.

- Describe the monoids are embeddable in a member of V.
- Describe the partially ordered monoids that are order embeddable in a member of V.
- Describe the lattice ordered monoids that are (order) embeddable in a member of V.

The low hanging fruits

Let **M** be a pomonoid. A **nucleus** γ on **M** is an increasing, monotone and idempotent map on M satisfying the further condition $\gamma(a)\gamma(b) \leq \gamma(ab)$. It is well known that the set $M_\gamma = \gamma(M)$ can be given a structure of pomonoid by setting $\gamma(a)\gamma(b) = \gamma(ab)$.

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Moreover for any pomonoid **M**, the power set $\mathcal{P}(M)$ can be made into a pomonoid by defining the so-called *complex product*:

$$XY = \{xy : x \in Y, y \in Y\}.$$

It is easily checked that if γ is a nucleus on $\mathcal{P}(M)$, then it is a closure operator on M. Thus $\mathcal{P}(M)_g$ coincide with the set the closed sets (i.e. those $X \subseteq M$ such that $\gamma(X) = X$), that therefore form a complete ℓ-monoid: the universe is $\gamma(\mathcal{P}(M))$, the meet is the intersection, the join is the closure of the union, the product is $X * Y = \gamma(\{xy : x \in X, y \in Y\}$ and the monoidal unit is $E = \gamma(\{1\})$. Moreover in $\gamma(\mathcal{P}(M))$ we define

$$X \backslash Y = \{a : X * \gamma(\{a\}) \subseteq Y\}$$
$$X / Y = \{a : \gamma(\{a\}) * X \subseteq Y\}.$$

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Lemma

Let $\mathbf{M}$ *be a pomonoid and* γ *a nucleus on* $\mathbf{M}$. *Then*

$$\mathcal{D}_\gamma(\mathbf{M})) = \langle \gamma(\mathcal{P}(M)), \vee, \wedge, *, \backslash, /, E \rangle$$

is a residuated lattice and $a \longmapsto \gamma(\{a\})$ *is an order embedding of* $\mathbf{M}$ *into* $\mathcal{D}_\gamma(\mathbf{M})$.

Proposition

Every pomonoid is order embeddable in a residuated lattice. Hence the class of ℓ-monoidal subreducts of residuated lattices is the variety of ℓ-monoids.

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Proposition

Any commutative pomonoid is order embeddable in a commutative residuated lattice. Hence the class of ℓ-monoidal subreducts of commutative residuated lattices is the variety of commutative ℓ-monoids.

The Dedekind MacNeille completion

If **M** is a pomonoid then $\downarrow X = \{b : b \leq a, \text{ for some } a \in X\}$ is a nucleus on $\mathcal{P}(M)$ which we call the **DM-nucleus**; in this case we drop the subscript γ and we call $\mathcal{D}(\mathbf{M})$ the **Dedekind-MacNeille completion** of **M**.

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Any monoid **M** is a pomonoid with the discrete ordering and its Dedekind McNeille completion is just $\mathcal{P}(\mathbf{M})$. Thus:

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Proposition

Every (commutative) monoid is embeddable in a (commutative), complete and distributive residuated lattice.

A residuated lattice or a pomonoid is **integral** if 1 is the largest element in the ordering. It is easy to see that if a monoid **M** is integral, then so is $\mathcal{D}(\mathbf{M})$ so

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A residuated lattice (or an ℓ-monoid) is **semilinear** if is a subdirect product of totally ordered residuated lattices (ℓ-monoids). Clearly any semilinear variety of residuated lattices (ℓ-monoids) consists entirely of distributive residuated lattices (ℓ-monoids). Let SCIRL the variety of semilinear commutative and integral residuated lattices.

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It is straightforward to check that the property of being totally ordered is preserved by the DM-nucleus.

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The class of ℓ-monoidal subreducts of SCIRL *is exactly the variety of distributive, commutative and integral ℓ-monoids.*

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Proof.

By an old result of Fuchs (1974) if $\mathbf{M}$ is a distributive commutative and integral ℓ-monoid then it is subdirectly embeddable in $\prod_{i \in I} \mathbf{M}_i$ where each $\mathbf{M}_i$ is totally ordered.

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By the above observation each $\mathbf{M}_i$ is embeddable in $\mathrm{D}(\mathbf{M}_i)$, a totally ordered commutative and integral residuated lattice. Hence $\mathbf{M}$ is an ℓ-monoidal subreduct of $\mathbf{B} = \prod_{i\in I}\mathrm{D}(\mathbf{M}_i)$. Clearly $\mathbf{B} \in \mathrm{SCIRL}$.

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On the other hand let $\mathbf{A}$ be an ℓ-monoidal subreduct of $\mathbf{B} \in \mathsf{SCIRL}$. Then $\mathbf{B}$ is subdirectly embeddable $\prod_{i \in I} \mathbf{B}_i$, where each $\mathbf{B}_i$ is totally ordered. Let $\mathbf{B}', \mathbf{B}'_i$ the ℓ-monoidal reducts of $\mathbf{B}, \mathbf{B}_i$, $i \in I$.

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Then certainly $\mathbf{B}'$ is embeddable in $\prod_{i \in I} \mathbf{B}'_i$ and so is $\mathbf{A}$. But the $\mathbf{B}'_i$ are totally ordered ℓ-monoids, so $\prod_{i \in I} \mathbf{B}'_i$ is a distributive commutative and integral ℓ-monoid and so is $\mathbf{A}$. □

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$$a \leq b \quad \text{if and only if} \quad \exists\, c \text{ such that } a = bc.$$

Then $\leq$ is always a preorder; let θ be the associated equivalence relation, and for $a \in M$ we denote by $\bar{a}$ the equivalence class of $a \in M$.

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Lemma

If $\mathbf{M}$ is a commutative monoid then θ is a congruence on $\mathbf{M}$ and $\mathbf{M}/\theta = \overline{\mathbf{M}}$ is an integral pomonoid with largest element $\bar{1}$.

A commutative monoid $\mathbf{M}$ is naturally ordered if θ is the trivial congruence, i.e. $\leq$ is a partial order. By the Lemma the partial order is compatible and $\mathbf{M}$ is a pomonoid;

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Lemma

For a commutative monoid $\mathbf{M}$ *the following are equivalent*

1. $\mathbf{M}$ *is naturally ordered;*
2. $\mathbf{M}$ *satisfies the quasiequation*

$$x \approx yz \ \& \ y \approx xw \quad \Rightarrow \quad x \approx y. \tag{N}$$

Hence naturally ordered commutative monoids form a quasivariety, denoted by NCM.

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Hence naturally ordered commutative monoids form a quasivariety, denoted by NCM.

Theorem

The class of monoidal subreducts of the variety of commutative and integral residuated lattices is exactly NCM.

ℓ-monoids and ℓ-groups

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A pomonoid (ℓ-monoid) $\mathbf{M}$ is **order cancellative** if, for any $a, b, c \in M$, $ab \leq ac$ or $ba \leq ca$ implies $b \leq c$. It is obvious that any order cancellative pomonoid is cancellative in the usual sense. The proof of the following *ordered* version of Steinitz's result is straightforward.

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Theorem

An abelian pomonoid $\mathbf{M}$ *is order embeddable in a pogroup if and only if it is order cancellative. In particular an abelian ℓ-monoid is order embeddable in an ℓ-group if and only if it is order cancellative.*

Negative cones

Remark

There is an abelian pomonoid $\mathbf{M}$ *that is cancellative but not order cancellative. Hence* $\mathbf{M}$ *is not embeddable in any pogroup but its monoidal reduct is embeddable in a group.*

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Negative cones of abelian ℓ-groups are really *cancellative hoops*, i.e. commutative, prelinear and order cancellative residuated lattices that are also divisible, i.e. satisfy the further equation $(x \to y)x \approx x \wedge y$. Let's denote this variety by C.

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Proposition

The class of pomonoidal (ℓ-monoidal) subreducts of C *is the class of all integral, abelian and order cancellative pomonoids (the quasivariety of all abelian and order cancellative ℓ-monoids).*

Example (Bahls-Galatos-Cole-Tsinakis 2003)

There is totally ordered, order cancellative, abelian and integral residuated lattice that is not embeddable in the negative cone of an ℓ-group but whose pomonoidal reduct is .

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Let **M** be the free abelian monoid generated by x and y; if we set a usual $x^0 = y^0 = 1$, then the elements of **M** are of the form $x^i y^j$, $i, j \in \mathbb{N}$. Let's order **M** in this way

$$1 > x > y > x^2 > xy > y^2 > x^3 > x^2y > xy^2 > y^3 > x^4 > \dots.$$

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Then **M** is an integral totally ordered abelian pomonoid that is also order cancellative, so it is order embeddable in the negative cone of an ℓ-group. However it is also a lattice and it can be made into a residuated lattice **L**, since the $\to$ is totally determined by $\vee$ and $\cdot$.

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However it cannot be embedded in the negative cone of an ℓ-group since it is not divisible:

$$(x \to y)x = x^2 \neq y = x \wedge y.$$

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Here is an example of a totally ordered commutative CIRL whose pomonoidal subreduct does not satisfy the quasiequation. For instance the 5-element CIRL ordered by $1 > a > b > c > d$ whose multiplication table is

	a	b	c	d	1
a	a	d	d	d	a
b	d	d	d	d	b
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Here $a = a^2$, $b \leq a$ but $ab = d < b$.

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- continue to study the combination of the power monoid and the nucleus construction which we have exploited only minimally; there is a sizable body of work involving nuclei that tackle questions that are tangential to ours;

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  We also have some more general results, too technical to explain here, which seem to hint to a complete solution of the problem.

THANK YOU!