On automorphisms of the endomorphism semigroup of the free monogenic dimonoid

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1.1. The problem of B.I. Plotkin

Let Θ be a variety of algebras and Θ^0 the category of all free in Θ algebras $W = W(X)$, where X is finite.

The main problem is to describe $Aut(\Theta^0)$ for a given Θ.

This problem about automorphisms of the monoid $End(A)$, for a free algebra A, was raised by B.I. Plotkin in his papers on universal algebraic geometry:

- Seven Lectures on the Universal Algebraic Geometry, Preprint, Inst. of Math., Hebrew University, 2000.
- Problems in algebra inspired by universal algebraic geometry, Fundam. Prikl. Mat., 2004, Vol. **10**, no. 3, 181–197.

1.2. The known results

For the variety of groups:

- E. Formanek, *A question of B. Plotkin about the semigroup of endomorphisms of a free group*, Proc. American Math. Society, **130** (2001), 935–937.

For varieties of semigroups and monoids:

- G. Mashevitsky, B.M. Schein, *Automorphisms of the endomorphism semigroup of a free monoid or a free semigroup*, Proc. AMS, Vol. **131** (2003), No. 6, 1655–1660.

For the variety of Lie algebras:

- G. Mashevitzky, B. Plotkin, E. Plotkin, *Automorphisms of the category of free Lie algebras*, J. Algebra, Vol. **282** (2004), 490–512.

1.2. The known results

For varieties of commutative and associative algebras:

- A. Berzins, *The group of automorphisms of the semigroup of endomorphisms of free commutative and free associative algebras*, Internat. J. Algebra Comput., Vol. **17** (2007), No. 5-6, 941–949.

For the variety of commutative dimonoids (g-dimonoids):

- Yu.V. Zhuchok, *Automorphisms of the endomorphism semigroup of a free commutative g-dimonoid*, Algebra Discrete Math., Vol. **21** (2016), No. 2, 309–324.
- Yu.V. Zhuchok, *Automorphisms of the endomorphism semigroup of a free commutative dimonoid*, Communic. Algebra, Vol. **45** (2017), No. 9, 3861–3871.

1.2. The known results

For the variety of dimonoids:

- Yu.V. Zhuchok, *Automorphisms of the category of free dimonoids*, J. Algebra, Vol. **657** (2024), no. 1, 883–895.

For the variety of nilpotent groups:

- A. Tsurkov., *Automorphisms of the category of the free nilpotent groups of the fixed class of nilpotency*, Internat. J. Algebra Comput., Vol. **17** (2007), No. 5-6, 1273–1281.

In general, for algebras of arbitrary varieties:

- G. Mashevitzky, B. Plotkin, E. Plotkin, *Automorphisms of categories of free algebras of varieties*, Electronic Research Announcements of the AMS, Vol. **8** (2002), 1–10.

2.1. The notion of a dimonoid

- J.-L. Loday, *Dialgebras and related operads*, Lect. Notes Math. 1763, Springer-Verlag, Berlin, 2001, 7–66.

Definition

An algebra $(D, \dashv, \vdash)$ with two binary associative operations $\dashv$ and $\vdash$ is called a *dimonoid* if for all $x, y, z \in D$,

$$\begin{array}{ll} (D_1) & (x \dashv y) \dashv z = x \dashv (y \vdash z), \\ (D_2) & (x \vdash y) \dashv z = x \vdash (y \dashv z), \\ (D_3) & (x \dashv y) \vdash z = x \vdash (y \vdash z). \end{array}$$

An element e of dimonoid $(D, \dashv, \vdash)$ is called a *bar-unit* if $e \dashv x = x = x \vdash e$ for all $x \in D$.

2.2. Relationships of dimonoids with other algebras

Semigroups

If operations of a dimonoid coincide, then it becomes a semigroup. Dimonoids have relationships with interassociativity for semigroups originated by Drouzy, strong interassociativity for semigroups introduced by Gould and Richardson, P-related semigroups considered by Hewitt and Zuckerman and *n*-tuple semigroups.

Dialgebras

Dialgebras are vector spaces over a field equipped with two binary associative operations satisfying the dimonoid axioms. With the help of properties of free dimonoids, free dialgebras were described and a cohomology of dialgebras was investigated by J.-L.Loday.

2.2. Relationships of dimonoids with other algebras

Duplexes and doppelalgebras

The notion of a doppelalgebra was considered by Richter in the context of algebraic K-theory. It is a vector space over a field equipped with two binary linear associative operations $\dashv$ and $\vdash$ satisfying the axioms

$$(x \vdash y) \dashv z = x \vdash (y \dashv z), \quad (x \dashv y) \vdash z = x \dashv (y \vdash z).$$

Genegalized dimonoids

g-dimonoid is a dimonoid without (D_2) $\quad (x \vdash y) \dashv z = x \vdash (y \dashv z)$.
Free g-dimonoids were constructed by Yu.Movsisyan, S.Davidov and M.Safaryan (2014).

2.2. Relationships of dimonoids with other algebras

Digroups

A nonempty set G equipped with two binary operations $\dashv$ and $\vdash$, a unary operation $^{-1}$, and a nullary operation 1, is called a *digroup* if the following conditions hold:

$$(G_1)\ (G, \dashv) \text{ and } (G, \vdash) \text{ are semigroups},$$

$$(G_2)\ x \vdash (x \dashv z) = (x \vdash x) \dashv z,$$

$$(G_3)\ 1 \vdash x = x = x \dashv 1,$$

$$(G_4)\ x \vdash x^{-1} = 1 = x^{-1} \dashv x.$$

An element x^{-1} is said to be *inverse* to x with respect to 1.

2.2. Relationships of dimonoids with other algebras

Trioids and trialgebras

A dimonoid $(D, \dashv, \vdash)$ with associative operation $\perp$ is *a trioid* if

$$\begin{aligned}
(x \dashv y) \dashv z &= x \dashv (y \perp z), \\
(x \perp y) \dashv z &= x \perp (y \dashv z), \\
(x \dashv y) \perp z &= x \perp (y \vdash z), \\
(x \vdash y) \perp z &= x \vdash (y \perp z), \\
(x \perp y) \vdash z &= x \vdash (y \vdash z).
\end{aligned}$$

An trialgebra is a vector space equipped with three binary associative operations satisfying the trioid axioms. Trialgebras appeared first in the paper of Loday and Ronco as a non-commutative version of Poisson algebras.

2.2. Relationships of dimonoids with other algebras

Leibniz algebras

A vector space L over a field with a binary operation $[,]$ is called *Leibniz algebra* if $[[a,b],c] = [a,[b,c]] - [b,[a,c]]$ for all $a,b,c \in L$.

It is well-known that for Lie algebras there is a notion of a universal enveloping associative algebra. Loday found a universal enveloping algebra for Leibniz algebras which are a non-commutative variation of Lie algebras. Dialgebras play a role of such object. The identities for dimonoids are chosen in such a way that the new operation

$$[x,y] = x \dashv y - y \vdash x$$

converts a dialgebra into a Leibniz algebra.

2.3. Examples of dimonoids

Example

Let X be an arbitrary nonempty nonsingleton set. Define

$$x \dashv y = x, \;\; x \vdash y = y.$$

Then $(X, \dashv, \vdash)$ is a dimonoid in which every element is a bar-unit.

Example

Let V be a finite dimensional vector space and $\varphi : V \to V$ be an idempotent linear operator. Define two operations $\dashv, \vdash$ on V by

$$x \dashv y = x\varphi + y, \; x \vdash y = x + y\varphi$$

for all $x, y \in V$. Then $(V, \dashv, \vdash)$ is a dimonoid.

2.4. The construction of a free dimonoid

Let X be an arbitrary set, $\overline{X} = \{\overline{x} \mid x \in X\}$ and

$$Y_n^{(1)} = \underbrace{\overline{X} \times X \times \ldots \times X}_{n}, \ldots, Y_n^{(n)} = \underbrace{X \times X \times \ldots \times \overline{X}}_{n}.$$

We denote the union of n copies $Y_n^{(i)}, 1 \leq i \leq n$, of X^n by Y_n and let $Fd(X) = \bigcup_{n \geq 1} Y_n$. Define operations $\dashv$ and $\vdash$ on $Fd(X)$ by

$$(x_1, ..., \overline{x_i}, ..., x_m) \dashv (y_1, ..., \overline{y_j}, ..., y_n) = (x_1, ..., \overline{x_i}, ..., x_m, y_1, ..., y_n),$$

$$(x_1, ..., \overline{x_i}, ..., x_m) \vdash (y_1, ..., \overline{y_j}, ..., y_n) = (x_1, ..., x_m, y_1, ..., \overline{y_j}, ..., y_n).$$

The algebra $\mathfrak{F}\mathfrak{d}_X = (Fd(X), \dashv, \vdash)$ is *a free dimonoid*.

2.5. An isomorphism (crossed isomorphism) of dimonoids

We use $x_1 \ldots \overline{x_i} \ldots x_k$ instead of $(x_1, \ldots, \overline{x_i}, \ldots, x_k) \in \mathfrak{F}\mathfrak{d}_X$. It is known, any $\omega = x_1 \ldots \overline{x_i} \ldots x_k \in Fd(X)$ can be uniquely represented in the canonical form as $\omega = \overline{x_1} \vdash \cdots \vdash \overline{x_i} \dashv \cdots \dashv \overline{x_k}$.

Definition

Let $\mathfrak{D}_1 = (D_1, \dashv_1, \vdash_1)$ and $\mathfrak{D}_2 = (D_2, \dashv_2, \vdash_2)$ be arbitrary dimonoids. A mapping $\varphi : D_1 \to D_2$ is called *a homomorphism (a crossed homomorphism)* of $\mathfrak{D}_1$ into $\mathfrak{D}_2$ if for all $x, y \in D_1$,

$$(x \dashv_1 y)\varphi = x\varphi \dashv_2 y\varphi, \ \ (x \vdash_1 y)\varphi = x\varphi \vdash_2 y\varphi$$

$$((x \dashv_1 y)\varphi = x\varphi \vdash_2 y\varphi, \ \ (x \vdash_1 y)\varphi = x\varphi \dashv_2 y\varphi).$$

3.1. The mirror types of permutations of $\mathfrak{Fd}_X$

Lemma

Let X and Y be arbitrary nonempty sets. Every bijection $\varphi : X \to Y$ induces an isomorphism ε_φ, a crossed anti-isomorphism ε^*_φ, and bijections ε'_φ and $\varepsilon^\circ_\varphi$ of the free dimonoid $\mathfrak{Fd}_X$ into the free dimonoid $\mathfrak{Fd}_Y$ such that for all $\omega = x_1 \ldots \overline{x_i} \ldots\ x_k \in Fd(X)$,

$$\begin{aligned} \omega\varepsilon_\varphi &= x_1\varphi \ldots \overline{x_i\varphi} \ldots x_k\varphi, \\ \omega\varepsilon^*_\varphi &= x_k\varphi \ldots \overline{x_i\varphi} \ldots x_1\varphi, \\ \omega\varepsilon'_\varphi &= x_1\varphi \ldots \overline{x_{k-i+1}\varphi} \ldots x_k\varphi, \\ \omega\varepsilon^\circ_\varphi &= x_k\varphi \ldots \overline{x_{k-i+1}\varphi} \ldots x_1\varphi. \end{aligned}$$

Remark. None of the mappings $\varepsilon'_\varphi, \varepsilon^\circ_\varphi$ of this lemma is either an isomorphism or a crossed anti-isomorphism of $\mathfrak{Fd}_X$ into $\mathfrak{Fd}_Y$.

3.2. The mirror types of automorphisms of $End(\mathfrak{F}\mathfrak{d}_X)$

Lemma

Let $\mathfrak{F}\mathfrak{d}_X$ and $\mathfrak{F}\mathfrak{d}_X$ be the free dimonoids on X and Y respectively, and ξ an isomorphism or a crossed anti-isomorphism of $\mathfrak{F}\mathfrak{d}_X$ into $\mathfrak{F}\mathfrak{d}_Y$ or $\xi \in \{\varepsilon'_\varphi, \varepsilon^\circ_\varphi\}$. Then the mapping

$$\Phi : f \mapsto f\Phi = \xi^{-1} f \xi, \;\; f \in End(\mathfrak{F}\mathfrak{d}_X)$$

is an isomorphism of $End(\mathfrak{F}\mathfrak{d}_X)$ into $End(\mathfrak{F}\mathfrak{d}_Y)$.

Let id_X be the identity transformation of X. By the lemma,

$$f\Phi_1 = (\varepsilon^*_{id_X})^{-1} f \varepsilon^*_{id_X}, \;\; f\Phi_2 = (\varepsilon'_{id_X})^{-1} f \varepsilon'_{id_X}, \;\; f\Phi_3 = (\varepsilon^\circ_{id_X})^{-1} f \varepsilon^\circ_{id_X}$$

for all $f \in End(\mathfrak{F}\mathfrak{d}_X)$ are automorphisms of $End(\mathfrak{F}\mathfrak{d}_X)$.

3.3. The Klein four-group of automorphisms

By K_4 we denote the Klein four-group, i.e., it's a direct product of two groups C_2, Φ_0 is the identity automorphism of $End(\mathfrak{Fd}_X)$.

Lemma

$G = \{\Phi_i \mid 0 \leq i \leq 3\}$ is a group with respect to the composition of permutations isomorphic to the Klein four-group K_4.

It is clear, $G =< \Phi_1, \Phi_2 \mid \Phi_1^2 = \Phi_2^2 = (\Phi_1\Phi_2)^2 = \Phi_0 > \cong K_4$. Let $\mathfrak{Fd}_X$ be the free dimonoid on X. Each endomorphism Φ of $\mathfrak{Fd}_X$ is uniquely determined by a mapping $\varphi : \overline{X} \to Fd(X)$. Really, to define Φ, it suffices for all $u = u_1 ... \overline{u_i} ... u_n \in Fd(X)$ to put

$$u\Phi = \overline{u_1}\varphi \vdash \cdots \vdash \overline{u_i}\varphi \dashv \cdots \dashv \overline{u_n}\varphi.$$

3.4. The constant endomorphisms of $\mathfrak{F}\mathfrak{d}_X$

Definition

Let $u \in Fd(X)$. An endomorphism $\theta_u \in End(\mathfrak{F}\mathfrak{d}_X)$ is called *constant* if $\overline{x}\theta_u = u$ for all $x \in X$.

Lemma

(i) An endomorphism f of $\mathfrak{F}\mathfrak{d}_X$ is constant if and only if $\psi f = f$ for all $\psi \in Aut(\mathfrak{F}\mathfrak{d}_X)$.
(ii) An endomorphism f of $\mathfrak{F}\mathfrak{d}_X$ is constant idempotent if and only if $f = \theta_{\overline{x}}$ for some $x \in X$.

Definition

An automorphism Ψ of the endomorphism monoid $End(\mathfrak{F}\mathfrak{d}_X)$ of the free dimonoid $\mathfrak{F}\mathfrak{d}_X$ is called *stable* if $\theta_{\overline{x}}\Psi = \theta_{\overline{x}}$ for all $x \in X$.

3.5. The stable automorphisms of $End(\mathfrak{F}\mathfrak{d}_X)$

Lemma

Let Ψ be a stable automorphism of $End(\mathfrak{F}\mathfrak{d}_X)$, $g \in End(\mathfrak{F}\mathfrak{d}_X)$ and $x \in X$. Then the following equalities hold:

(i) $\theta_u \Psi = \theta_v$ *implies* $c(u) = c(v)$;
(ii) $|\overline{x}g| = |\overline{x}(g\Psi)|$.

Corollary

Let Ψ be a stable automorphism of the endomorphism monoid $End(\mathfrak{F}\mathfrak{d}_X)$ and $x_1, x_2 \in X$ are distinct. Then

$$\theta_{\overline{x_1}x_2}\Psi \in \{\theta_{\overline{x_1}x_2}, \theta_{x_1\overline{x_2}}, \theta_{\overline{x_2}x_1}, \theta_{x_2\overline{x_1}}\}.$$

3.5. The stable automorphisms of $End(\mathfrak{F}\mathfrak{d}_X)$

Lemma

Let Ψ be a stable automorphism of the endomorphism monoid $End(\mathfrak{F}\mathfrak{d}_X)$ and $x_1, x_2 \in X$ are distinct. Then

(i) $\theta_{\overline{x_1}x_2}\Psi = \theta_{\overline{x_1}x_2}$ *and* $\theta_{x_1\overline{x_2}}\Psi = \theta_{x_1\overline{x_2}}$ *implies that* $\Psi = \Phi_0$*;*
(ii) $\theta_{\overline{x_1}x_2}\Psi = \theta_{x_2\overline{x_1}}$ *and* $\theta_{x_1\overline{x_2}}\Psi = \theta_{\overline{x_2}x_1}$ *implies that* $\Psi = \Phi_1$*;*
(iii) $\theta_{\overline{x_1}x_2}\Psi = \theta_{x_1\overline{x_2}}$ *and* $\theta_{x_1\overline{x_2}}\Psi = \theta_{\overline{x_1}x_2}$ *implies that* $\Psi = \Phi_2$*;*
(iv) $\theta_{\overline{x_1}x_2}\Psi = \theta_{\overline{x_2}x_1}$ *and* $\theta_{x_1\overline{x_2}}\Psi = \theta_{x_2\overline{x_1}}$ *implies that* $\Psi = \Phi_3$*.*

Lemma

Do not exist stable automorphisms of the endomorphism monoid $End(\mathfrak{F}\mathfrak{d}_X)$ distinct from Φ_i, where $0 \leq i \leq 3$.

4.1. End-perfect and End-semiperfect algebras

Definition

For an algebra A, an automorphism $\Phi : End(A) \to End(A)$ is *quasi-inner* if there exists a permutation $\alpha \in S(A)$ such that $\beta\Phi = \alpha^{-1}\beta\alpha$ for all $\beta \in End(A)$. If $\alpha \in Aut(A)$, then Φ is *inner*.

Definition

A free algebra $F \in \Theta$ is called *End-perfect* (*End-semiperfect*) [2] if every automorphism of $End(F)$ is inner (respectively, quasi-inner).

Example

For example, the free groups are End-perfect, and the free semigroups and the free monoids are End-semiperfect.

4.2. End-semiperfectness of free dimonoids

Theorem

(Zhuchok Yu., 2024) Let X be an arbitrary set with $|X| \geq 2$. Every isomorphism $\Phi : End(\mathfrak{F}\mathfrak{d}_X) \to End(\mathfrak{F}\mathfrak{d}_Y)$ is induced either by the isomorphism ε_f or by the crossed anti-isomorphism ε_f^*, or the bijections ε_f', ε_f° of $\mathfrak{F}\mathfrak{d}_X$ into $\mathfrak{F}\mathfrak{d}_Y$ for a uniquely determined bijection $f : X \to Y$.

Corollary

The free dimonoids $\mathfrak{F}\mathfrak{d}_X$ are End-semiperfect algebras.

The permutations of the monoid $End(\mathfrak{F}\mathfrak{d}_X)$ defined by $\eta E_f = \varepsilon_f^{-1}\eta\varepsilon_f$, $\eta(\Phi_i E_f) = (\varepsilon^\alpha_{id_X}\varepsilon_f)^{-1}\eta(\varepsilon^\alpha_{id_X}\varepsilon_f)$, where $f \in S(X)$, $i \in \{1,2,3\}$, $\alpha \in \{*, ', \circ\}$, are quasi-inner automorphisms.

4.3. The automorphism group of $End(\mathfrak{F}\mathfrak{d}_X)$, $|X| \geq 2$

Theorem

(Zhuchok Yu., 2024) The group $Aut(End(\mathfrak{F}\mathfrak{d}_X))$, $|X| \geq 2$, is isomorphic to the direct product of the Klein four-group K_4 and the symmetric group $S(X)$.

Definition

For an arbitrary algebra $\mathfrak{A}$, the quotient-group of $Aut(End(\mathfrak{A}))$ by the inner automorphism group $Inn(End(\mathfrak{A}))$ is called *outer automorphism group* and it is denoted by $Out(End(\mathfrak{A}))$.

Corollary

$Out(End(\mathfrak{F}\mathfrak{d}_X))$ is isomorphic to the Klein four-group K_4.

5.1. The free monogenic dimonoid

Proposition

(Zhuchok A., 2011) Let N be the set of all natural numbers. Define on the set $S = \{(a; b) \in N \times N \mid a \geq b\}$ two binary associative operations $\prec$ and $\succ$ as follows:

$$(a; b) \prec (c; d) = (a + c; b),$$

$$(a; b) \succ (c; d) = (a + c; a + d).$$

Then the free dimonoid $(Fd_1, \dashv, \vdash)$ of rank 1 is isomorphic to the algebra $(S, \prec, \succ)$.

5.2. Endomorphisms of a free dimonoid of rank 1

Theorem

(Zhuchok Yu., 2014) For any $(k; l) \in S$ *a transformation* $\xi_{k,l}$ *of the free dimonoid* $(Fd_1, \dashv, \vdash)$ *defined by* $(a; b)\xi_{k,l} = (ak; (b-1)k + l)$ *is a monomorphism. And every endomorphism of* $(Fd_1, \dashv, \vdash)$ *has the above form.*

Consider a binary operation $\circ$ on S defined as follows:

$$(a; b) \circ (c; d) = (ac; (b-1)c + d).$$

Theorem

(Zhuchok Yu., 2014) The endomorphism semigroup $End(Fd_1)$ *of the free dimonoid* $(Fd_1, \dashv, \vdash)$ *is isomorphic to the semigroup* $(S, \circ)$.

5.3. Some properties of $(S, \circ)$

For every $a \in N$, let $S_a = \{(a, k) \in S | k \leq a\}$.

Lemma

Define a relation ρ on the semigroup $(S, \circ)$ by the rule $(a, b)\rho(c, d) \Leftrightarrow a = c$. Then ρ is a congruence and the quotient $(S, \circ)/\rho \cong (N, \cdot)$.

It is known that $Aut(N, \cdot)$ is isomorphic to the symmetric group $S(P)$ defined on the set P of all prime numbers.

Corollary

Let Φ be an arbitrary automorphism of the semigroup $(S, \circ)$. Then for any $a \in N$, we have $\Phi(S_a) = S_a$.

5.4. Automorphisms of $(S, \circ)$

Let Φ_0 be the identity automorphism of the semigroup $(S, \circ)$.

Lemma

Define a transformation Φ_1 of the semigroup $(S, \circ)$ by the rule $\Phi_1(a, b) = (a, a - b + 1)$. Then Φ_1 is an automorphism of the semigroup $(S, \circ)$.

Theorem

Let Φ be an arbitrary automorphism of the semigroup $(S, \circ)$. Then
(i) $\Phi(2, 1) = (2, 1)$ implies that $\Phi = \Phi_0$,
(ii) $\Phi(2, 1) = (2, 2)$ implies that $\Phi = \Phi_1$.

5.5. The automorphism group of $End(\mathfrak{Fd}_X)$

Let K_4 be the Klein four-group, C_2 is a two-element group.

Theorem

Automorphism group of the endomorphisms semigroup of the free dimonoid $\mathfrak{Fd}_X$ is isomorphic to the group C_2 if X is singleton, and it is isomorphic to the direct product of the Klein four-group K_4 with the symmetric group $S(X)$ if $|X| \geq 2$, i.e.

$$Aut(End(\mathfrak{Fd}_X)) \cong \begin{cases} C_2, & |X| = 1, \\ K_4 \times S(X), & |X| \geq 2. \end{cases}$$

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THANK YOU FOR ATTENTION!