On the difference graph of a group

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Overview

We associate a graph with a group or an algebra

- What does the graph tell about the algebra?
- Determine the relation between the graph and the algebra.
- Describe the graph.
- **What properties does the graph have?**
- **Determine the relation between different associated graphs.**

Overview

Some of the graphs associated to groups

- Cayley graph
- commuting graph
- directed power graph
- power graph
- enhanced power graph

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- **difference graph**

Definitions

“Definition”

Cayley graph *of a group...*

- *with directed on undirected edges...*
- *with labeled or unlabeled edges...*

The edge set may vary as well.

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Definition

The commuting graph *of a group* $\mathbf{G}$ *is the simple graph* $\mathcal{C}(\mathbf{G})$ *with vertex set* $X \subseteq G$ *such that* $x \overset{c}{\sim} y$ *if* $xy = yx$*.*

Definitions

Definition

The directed power graph *of a group* $\mathbf{G}$ *is the simple digraph* $\vec{\mathcal{G}}(\mathbf{G})$ *with vertex set* G *such that* $x \to y$ *if* $y \in \langle x \rangle$.
The power graph *of a group* $\mathbf{G}$ *is the simple graph* $\mathcal{G}(\mathbf{G})$ *with vertex set* G *such that* $x \sim y$ *if* $y \in \langle x \rangle$ *or* $x \in \langle y \rangle$.
The enhanced power graph *of a group* $\mathbf{G}$ *is the simple graph* $\mathcal{G}_e(\mathbf{G})$ *with vertex set* G *such that* $x \stackrel{e}{\sim} y$ *if* $\exists z \in G \;\; x, y \in \langle z \rangle$.

$$\mathcal{G}(\mathbf{G}) \subseteq \mathcal{G}_e(\mathbf{G}) \subseteq \mathcal{C}(\mathbf{G})$$

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$$\mathcal{G}(\mathbf{G}) \subseteq \mathcal{G}_e(\mathbf{G}) \subseteq \mathcal{C}(\mathbf{G})$$

The difference graph *of a group is* $\mathcal{G}_e(\mathbf{G}) \setminus \mathbf{G}(\mathbf{G})$ *with isolated vertices removed.*

What we want to know

- What are its cliques? When is it complete?
- Forbidden subgraphs?
- Diameter and dominating sets?
- **When does it determine the group?**
- **When do different graphs determine each other?**

Other questions regarding its:

- chromatic number
- clique number
- perfectness
- independence number
- automorphism group

Introduction of these graphs

A. V. Kelarev, S. J. Quinn, 2000

- introduced the directed power graph

I. Chakrabarty, S. Ghosh, M. K. Sen, 2009

- the power graph first studied

G. Aalipour, S. Akbari, P. J. Cameron, R. Nikandish, F. Shaveisi, 2017

- introduced the enhanced power graph

Isomorphism problem

Theorem (Cameron, Ghosh, 2011)

Let $\mathbf{G}$ *and* $\mathbf{H}$ *be finite abelian groups. Then* $\mathcal{G}(\mathbf{G}) \cong \mathcal{G}(\mathbf{H}) \Rightarrow \mathbf{G} \cong \mathbf{H}$.

Theorem (Cameron, 2010)

Let $\mathbf{G}$ *and* $\mathbf{H}$ *be finite groups. Then* $\mathcal{G}(\mathbf{G}) \cong \mathcal{G}(\mathbf{H}) \Rightarrow \vec{\mathcal{G}}(\mathbf{G}) \cong \vec{\mathcal{G}}(\mathbf{H})$.

Isomorphism problem

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Corollary

Let $\mathbf{G}$ *and* $\mathbf{H}$ *be finite groups. Then*

$$\vec{\mathcal{G}}(\mathbf{G}) \cong \vec{\mathcal{G}}(\mathbf{H}) \quad \Leftrightarrow \quad \mathcal{G}(\mathbf{G}) \cong \mathcal{G}(\mathbf{H}) \quad \Leftrightarrow \quad \mathcal{G}_e(\mathbf{G}) \cong \mathcal{G}_e(\mathbf{H}).$$

Isomorphism problem

Isomorphism problem

Isomorphism problem
...on infinite groups!

Theorem (Cameron, Guerra, Jurina, 2019)

Let $\mathbf{G}$ *and* $\mathbf{H}$ *be nilpotency class* 2 *torsion-free groups. Then*

$$\mathcal{G}(\mathbf{G}) \cong \mathcal{G}(\mathbf{H}) \quad \Rightarrow \quad \vec{\mathcal{G}}(\mathbf{G}) \cong \vec{\mathcal{G}}(\mathbf{H}).$$

It works with the weaker assumption, too!

Isomorphism problem on infinite groups

Theorem (Z., 2022 & 2021)

Let $\mathbf{G}$ *and* $\mathbf{H}$ *be groups. If* $\mathbf{G}$ *is torsion-free, then*

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Theorem (Z., 2021)

Let $\mathbf{G}$ *and* $\mathbf{H}$ *be groups. If* $\mathbf{G}$ *is has no quasicyclic subgroup, then*

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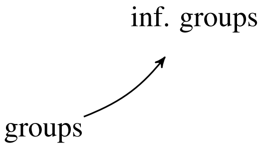
- It works for power-associative loops, too!

Why loops?

Why loops?



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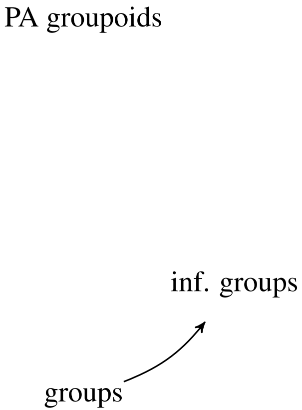

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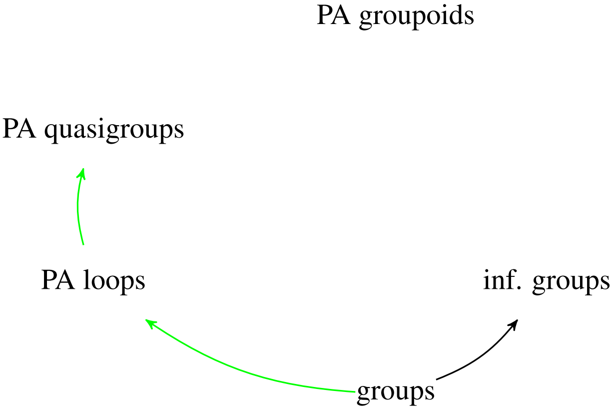

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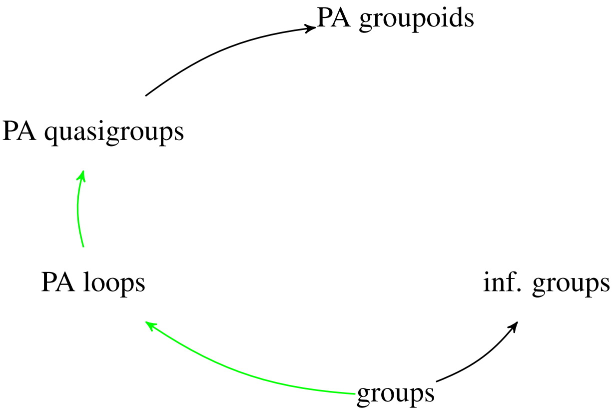

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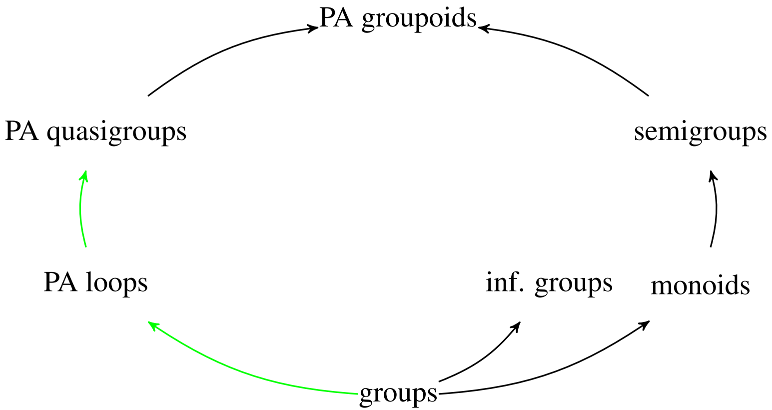
PA groupoids
PA quasigroups
semigroups
PA loops
inf. groups
monoids
groups

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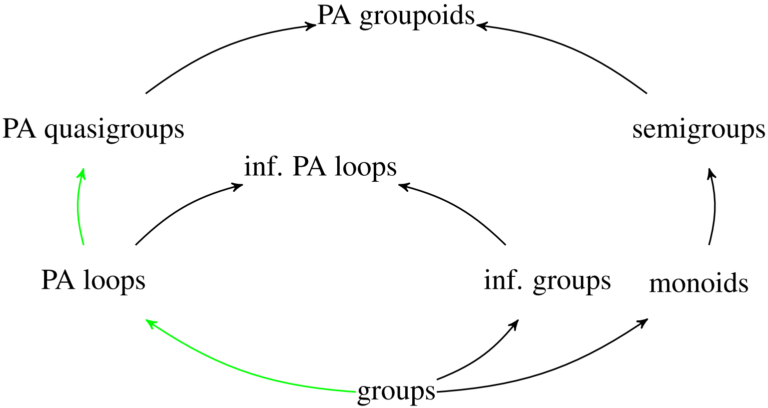
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Isomorphism problem on infinite groups

Theorem (Jafari, Z., 2023)

Let $\mathbf{G}$ *and* $\mathbf{H}$ *be nilpotent groups non of which is a Prüfer group. Then*

$$\mathcal{G}(\mathbf{G}) \cong \mathcal{G}(\mathbf{H}) \quad \Rightarrow \quad \vec{\mathcal{G}}(\mathbf{G}) \cong \vec{\mathcal{G}}(\mathbf{H}).$$

Theorem (Bošnjak, Madarász, Z., 2024)

Let $\mathbf{G}$ *and* $\mathbf{H}$ *be any groups. Then*

$$\mathcal{G}(\mathbf{G}) \cong \mathcal{G}(\mathbf{H}) \quad \Rightarrow \quad \mathcal{G}_e(\mathbf{G}) \cong \mathcal{G}_e(\mathbf{H}).$$

Back to finite!

We should be able to prove that $\mathcal{D}(\mathbf{G}) \cong \mathcal{D}(\mathbf{H}) \Rightarrow \mathcal{G}(\mathbf{G}) \cong \mathcal{G}(\mathbf{H})$!

Isomorphism problem

- $\mathcal{D}(\mathbf{D}_6) \cong \mathcal{D}(\mathbf{Q}_{12})$

Therefore, $\mathcal{D}(\mathbf{G}) \cong \mathcal{D}(\mathbf{H}) \not\Rightarrow \mathcal{G}(\mathbf{G}) \cong \mathcal{G}(\mathbf{H})$.

Theorem

Let $\mathbf{G}$ *and* $\mathbf{H}$ *be finite abelian groups whose orders have exactly two prime divisors. Then*

$$\mathcal{D}(\mathbf{G}) \cong \mathcal{D}(\mathbf{H}) \Rightarrow \mathbf{G} \cong \mathbf{H}.$$

Diameter

Theorem (Biswas, Cameron, Das, Dey, 2024)

If $\mathbf{G}$ *is a finite non-p-group with nontrivial center, then*

$$\operatorname{diam}(\mathcal{D}(\mathbf{D})) \leq 6.$$

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Theorem (Biswas, Cameron, Das, Dey, 2024)

If $\mathbf{G}$ *is a finite nilpotent non-p-group, then*

$$\operatorname{diam}(\mathcal{D}(\mathbf{D})) \leq 4.$$

Diameter

Theorem (Ma, Z., Žigerović, 2026+)

Let $\mathbf{G}$ *be a finite nilpotent non-p-group. Then,*

$$\operatorname{diam}(\mathcal{D}(G)) = \begin{cases} 2, & \textit{if } \exp(\mathbf{G}) \textit{ is a square-free number;} \\ 4, & \textit{if } \mathbf{G} \textit{ is a } \Psi\textit{-group;} \\ 3, & \textit{otherwise.} \end{cases}$$

(Ψ-groups defined in the paper.)

Corollary (Ma, Z., Žigerović, 2026+)

Let G *be an abelian non-p-group. Then*

$$\operatorname{diam}(\mathcal{D}(\mathbf{G})) = \begin{cases} 2, & \textit{if } \exp(G) \textit{ is a square-free number;} \\ 4, & \textit{if } \mathbf{C}_{p^2} \times \mathbf{C}_{p^2} \leq \mathbf{G}; \\ 3, & \textit{otherwise.} \end{cases}$$