Abelian congruences in varieties with a weak difference term

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Prologue

Motivation: finite algebras with a Taylor term (and their relations).

- *idempotent*: $t(x, x, \dots, x) = x$.
- enough identities of form $t(vars) = t(vars)$ implying it is not a projection.

Characterization Thm (Hobby, McKenzie, Maróti, Barto, Kozik, Siggers . . .)

Let $\mathbf{A}$ be a finite algebra. The following are equivalent:

1. $\mathbf{A}$ has a **Taylor** term.
2. $\mathbf{A}$ has an idempotent **weak near unanimity** term (WNU):
$$w(y, x, \dots, x) = w(x, y, \dots, x) = \cdots = w(x, \dots, x, y).$$
3. (Idempotent cyclic term, Siggers term, Zhuk term . . .)
4. $\mathsf{HSP}(\mathbf{A})$ has a **weak difference term**.

This lecture is an advertisement for weak difference terms.

Part 1: Definitions

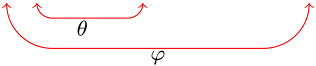
Centrality, abelian

Definition. Let $\varphi, \theta \in \mathrm{Con}(\mathbf{A})$. We say that φ **centralizes** θ, and write

$$C(\varphi, \theta; 0) \quad \text{or} \quad [\varphi, \theta] = 0,$$

if $\forall$ $(m+n)$-ary term $t(\mathbf{x}, \mathbf{y})$, $\forall\,(a_i, b_i) \in \varphi$, $\forall\,(c_j, d_j) \in \theta$,

$$t(\mathbf{a}, \mathbf{c}) = t(\mathbf{a}, \mathbf{d}) \implies t(\mathbf{b}, \mathbf{c}) = t(\mathbf{b}, \mathbf{d}).$$



θ is **abelian** iff $[\theta, \theta] = 0$.

A is **abelian** iff $[1, 1] = 0$.

The **centralizer** of θ is the $\underbrace{\text{largest } \varphi \in \mathrm{Con}(\mathbf{A}) \text{ satisfying } [\varphi, \theta] = 0.}_{(0\,:\,\theta)}$

Theorem (Gumm, Herrmann 1979)

Suppose **A** is abelian and has a *Mal'tsev* term $p(x, y, z)$:

$$p(x, x, y) = y = p(y, x, x).$$

Fix any $e \in A$, and define

$$x + y := p(x, e, y).$$

Then:

1. $+$ is an abelian group operation on A, with identity element e.
2. $p(x, y, z) = x - y + z$.
3. Every $f \in \mathrm{Pol}_n(\mathbf{A})$ is affine w.r.t. $(A, +)$:

$$f(x_1, \ldots, x_n) = \left(\sum_{i=1}^{n} r_i(x_i)\right) + c$$

for some $r_i \in \mathrm{End}(A, +)$ and $c \in A$.

(Hence **A** is polynomially equivalent to a module.)

($\mathbf{A}$ abelian, $p(x,x,y) = y = p(y,x,x)$, $e \in A$, $x + y := p(x,e,y)$)

Proof of $x + y = y + x$

Let $\quad t(x, y_1, y_2) := p(y_1, x, y_2)$. $\qquad$ Let $a, b \in A$.

Because $\mathbf{A}$ is abelian, i.e., $[1,1] = 0$, and $(a,e), (a,b), (b,a) \in 1$, we have

$$t(\underline{a}, \underbrace{a, b}_{\mathbf{c}}) = t(\underline{a}, \underbrace{b, a}_{\mathbf{d}}) \quad \Longrightarrow \quad t(\underline{e}, \underbrace{a, b}_{\mathbf{c}}) = t(\underline{e}, \underbrace{b, a}_{\mathbf{d}}).$$

($\mathbf{c} \overset{1}{\longleftrightarrow} \mathbf{d}$; $\; a \overset{1}{\longleftrightarrow} e$)

I.e.,

$$p(a, \underline{a}, b) = p(b, \underline{a}, a) \quad \Longrightarrow \quad \underbrace{p(a, \underline{e}, b)}_{a+b} = \underbrace{p(b, \underline{e}, a)}_{b+a}.$$

LHS is true (by Mal'tsev identities). $\qquad$ Hence RHS is also true. $\square$

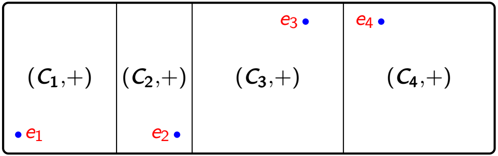
There is an analogous theorem for abelian congruences.

Folklore theorem for abelian congruences

Suppose $\theta \in \mathrm{Con}(\mathbf{A})$ is abelian, and $\mathbf{A}$ has an idempotent term $d(x,y,z)$ whose restriction $d{\restriction}_C$ to each θ-class C is Mal'tsev.[1]

Then:

1. Each θ-class C can be turned into an abelian group (same recipe): pick $e \in C$, define $x + y := d(x,e,y)$.
2. The restriction of any polynomial to an n-tuple of θ-classes is affine with respect to the abelian groups on the classes and the target class.

$A =$ 

(θ-classes)

[1] I.e., $d(x,x,y) = y = d(y,x,x)$ whenever $x \overset{\theta}{\equiv} y$.

(Generalizing Gumm's "difference term" for congruence modular varieties):

Definition (Kearnes 1995, Lipparini 1996)

Let $d(x, y, z)$ be a 3-ary term.

We say that $d(x, y, z)$ is a **weak difference** term (or **WD** term) ...

- ...for an algebra **A** if it is idempotent and its restriction to any class of any abelian congruence of **A** is Mal'tsev.[1]
- ...for a variety if it is a WD term for every algebra in the variety.

Note: this definition is not in terms of identities.

[1] The "real" definition requires that this property hold not only for **A**, but also for every homomorphic image of **A**.

Having a WD term gives us:

1. Abelian groups on classes of abelian congruences (Folklore Theorem).
2. Some technical properties of abelian congruences, similar to the congruence modular case, such as:

Technical Lemma

Suppose **A** belongs to a variety with a WD term, $\theta \in \mathrm{Con}(\mathbf{A})$, θ is abelian.

1. $\theta \vee \delta = \delta \circ \theta \circ \delta$ for all $\delta \in \mathrm{Con}(\mathbf{A})$.
2. $C(\theta, \theta; \delta)$ for all $\delta \leq \theta$.
3. $(a, b) \in \theta \implies \mathrm{Cg}(a, b) = \{(f(a), f(b)) : f \in \mathrm{Pol}_1(\mathbf{A})\}$.

The proofs are elementary. For example:

(1). We must show $\theta \circ \delta \circ \theta \subseteq \delta \circ \theta \circ \delta$. Assume $a \overset{\theta}{\equiv} x \overset{\delta}{\equiv} y \overset{\theta}{\equiv} b$. Then

$$a = d(a, x, x) \overset{\delta}{\equiv} d(a, y, y) \overset{\theta}{\equiv} d(x, y, b) \overset{\delta}{\equiv} d(y, y, b) = b.$$

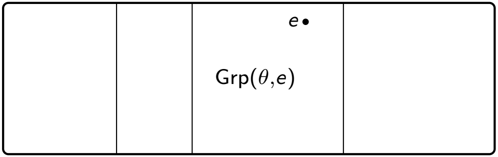
Part 2: Focus on the abelian groups

Notation

Assume $d(x, y, z)$ is a WD term for **A**.

Let $\theta \in \mathrm{Con}(\mathbf{A})$ be abelian, $e \in A$, and $C = e/\theta$.

$A =$



(θ-classes)

<u>Notation</u>:

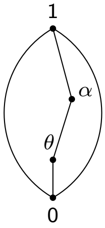
$$\boxed{\mathrm{Grp}(\theta, e)} \text{ denotes } (C, +) \text{ where } x + y := d(x, e, y).$$

1980s notation: $M(\theta, e)$

(**A** has a WD term, θ is abelian)

Question: how are groups on <u>different</u> θ-classes related?

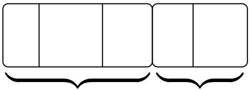
Con **A** =



Let $\alpha = (0 : \theta)$.

Group θ-classes together if they are in the same α-class.

$A =$



(θ-classes)

α-classes

Theorem (Gumm 1983)

If **A** belongs to a congruence modular variety, then

$$\mathrm{Grp}(\theta, e) \cong \mathrm{Grp}(\theta, e') \quad \text{whenever} \quad e \stackrel{\alpha}{\equiv} e'.$$

How does Gumm's result generalize (if at all) to WD term varieties?

Answer #1:

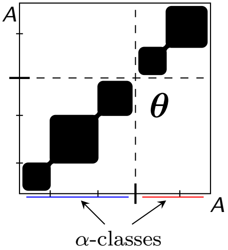
It doesn't generalize.

Answer #2: Sing, muse, a story from ancient times . . .

Assume **A** is in a variety with a WD term, θ is abelian, $\alpha = (0 : \theta)$.

(Following Hagemann & Herrmann 1979):

- Let $\boldsymbol{\theta}$:= the subalgebra of $\mathbf{A}^2$ with universe θ.

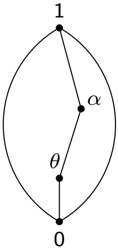


$\overline{\alpha}$-classes

- $\boldsymbol{\theta}$ itself has congruences, including: η_1 (kernel of 1st projection), η_2, $\overline{\theta}$ (classes are squares of θ-classes), $\overline{\alpha}$

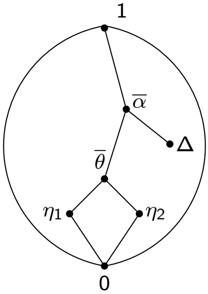
(θ abelian, $\alpha := (0 : \theta)$, in a variety with a WD term)

$\mathrm{Con}(\mathbf{A}) =$



$\mathrm{Con}(\boldsymbol{\theta}) =$

η_i:=ker(proj$_i$)



Easy Fact: $\overline{\theta}$ is abelian.

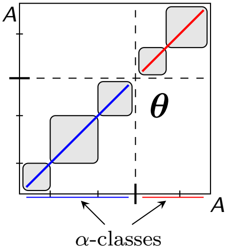
(Hagemann & Herrmann): $[\alpha, \theta] = 0 \iff \boldsymbol{\theta}$ has a congruence $\Delta \leq \overline{\alpha}$ where for each α-class E, the diagonal 0_E is a Δ-class.

Answer #2: Sing a story from ancient times ...

Assume **A** is in a variety with a WD term, θ is abelian, $\alpha = (0 : \theta)$.

(Following Hagemann & Herrmann 1979):

- Let $\boldsymbol{\theta}$:= the subalgebra of $\mathbf{A}^2$ with universe θ.



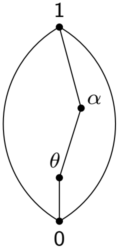
Δ-classes

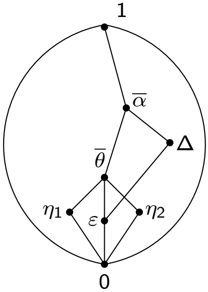
- $\boldsymbol{\theta}$ itself has congruences, including: η_1 (kernel of 1st projection), η_2, $\overline{\theta}$ (classes are squares of θ-classes), $\overline{\alpha}$, and $\Delta \leq \overline{\alpha}$ where

$$E \text{ an } \alpha\text{-class} \implies 0_E \text{ a } \Delta\text{-class.}$$

- Intuition: $(a, b) \stackrel{\Delta}{\equiv} (a', b')$ should mean $a - b$ "=" $a' - b'$.

(θ abelian, $\alpha := (0 : \theta)$, in a variety with a WD term)

$\text{Con}(\mathbf{A}) =$ 

$\text{Con}(\boldsymbol{\theta}) =$ 

$\eta_i := \ker(\text{proj}_i)$

Easy Fact: $\overline{\theta}$ is abelian.

(Hagemann & Herrmann): $[\alpha, \theta] = 0 \iff \boldsymbol{\theta}$ has a congruence $\Delta \leq \overline{\alpha}$ where for each α-class E, the diagonal 0_E is a Δ-class.

Clearly $\overline{\theta} \vee \Delta = \overline{\alpha}$. Let $\varepsilon := \overline{\theta} \wedge \Delta$ (see picture).

Technical Lemma $\implies C(\overline{\theta}, \overline{\theta}; \varepsilon)$ and $\overline{\alpha} = \Delta \circ \overline{\theta} \circ \Delta \implies \boxed{C(\overline{\alpha}, \overline{\alpha}; \Delta)}$

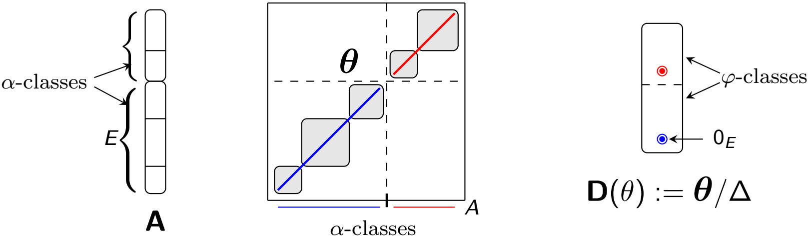
Conclusion: In $\boldsymbol{\theta}/\Delta$, $\overline{\alpha}/\Delta$ is an abelian congruence.

(θ abelian, $\alpha := (0 : \theta)$, in a variety with a WD term)

(Following Freese 1983): Define

$\mathbf{D}(\theta) := \boldsymbol{\theta}/\Delta$ the "**difference algebra** for θ"

$\varphi := \overline{\alpha}/\Delta$ the "**derived congruence**" (abelian)

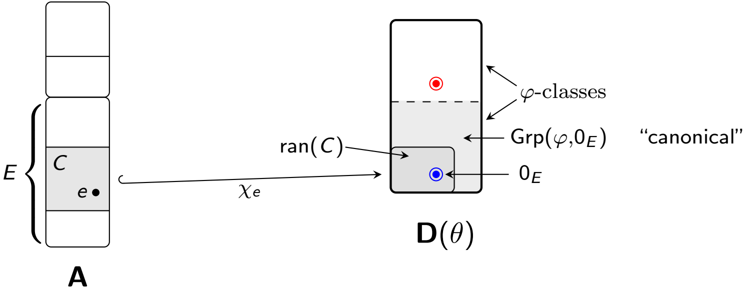


This wisdom of the ancients (known previously in congruence modular varieties) works in any variety with a WD term.

Application # 1: coherent abelian groups

Fix an α-class E.

For each θ-class $C \subseteq E$, define $\boxed{\mathrm{ran}(C) := C^2/\Delta}$, the "**range** of C."



Given $e \in C$, define $\chi_e : C \to \mathrm{ran}(C)$ by $\boxed{x \mapsto (x,e)/\Delta.}$

Embedding Lemma

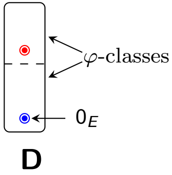
1. χ_e is an embedding $\mathrm{Grp}(\theta, e) \hookrightarrow \mathrm{Grp}(\varphi, 0_E)$ whose range is $\mathrm{ran}(C)$.
2. $\forall (a,b) \in C^2$, $(a,b)/\Delta = \chi_e(a) - \chi_e(b)$ in $\mathrm{Grp}(\varphi, 0_E)$.
3. The set of ranges of θ-classes in E is a directed set of subgroups of $\mathrm{Grp}(\varphi, 0_E)$ whose union is $\mathrm{Grp}(\varphi, 0_E)$.

Application #2:
From abelian groups to vector spaces

Let $T_0 := \{0_E \,:\, E \text{ an } \alpha\text{-class}\}$.

T_0 is a transversal for φ, and $T_0 \leq \mathbf{D}$.

Let $O : \mathbf{D} \to \mathbf{T}_0$ be the corresponding retraction.



Define:

$$F := \{\lambda \in \mathrm{End}(\mathbf{D}) \,:\, \lambda(\varphi) \subseteq \varphi \text{ and } \varphi|_{T_0} = \mathrm{id}_{T_0}\}.$$

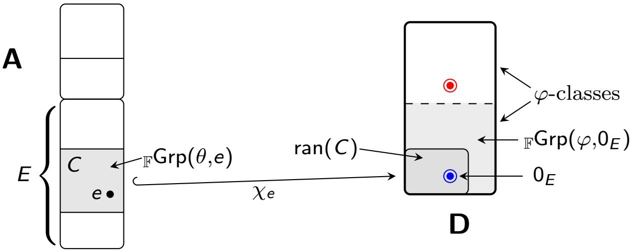
Define $+$ on F by

$$(\lambda + \mu)(x) := d(\lambda(x), O(x), \mu(x)).$$

Theorem

1. $\lambda, \mu \in F \implies \lambda + \mu \in F$.
2. $\mathbb{F} := (F, +, \circ, O, \mathrm{id}_D)$ is a unital ring, which acts naturally on each group $\mathrm{Grp}(\varphi, 0_E)$, making it a left $\mathbb{F}$-module ${}_{\mathbb{F}}\mathrm{Grp}(\varphi, 0_E)$.

(**A** in a variety with a WD term, θ abelian, $\alpha := (0 : \theta)$, $\mathbf{D} := \boldsymbol{\theta}/\Delta$, $\varphi := \overline{\alpha}/\Delta$)



Theorem

Assume additionally that $0 \prec \theta$. Then:

1. $\mathbb{F}$ is a division ring. (Hence ${}_{\mathbb{F}}\mathrm{Grp}(\varphi, 0_E)$ is a vector space.)
2. For each θ-class $C \subseteq E \in A/\alpha$, ran($C$) is a sub<u>space</u> of ${}_{\mathbb{F}}\mathrm{Grp}(\varphi, 0_E)$.

Via χ_e, each θ-class C naturally inherits an $\mathbb{F}$-vector space structure!

Payoff: rectangular critical relations (if enough time)

Let $R \leq_{sd} \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ be a subdirect relation of finite algebras $\mathbf{A}_i$.

1. R is **critical** if it is
   - meet-irreducible in the lattice of subuniverses of $\mathbf{A}_1 \times \cdots \times \mathbf{A}_n$, and
   - indecomposable, i.e., $R \neq \mathrm{proj}_I(R) \times \mathrm{proj}_{[n]\setminus I}(R)$ for all $\varnothing \neq I \subset [n]$.
2. R is **rectangular** if (blah blah blah).
3. R is **completely functional** if each coordinate is determined by the remaining coordinates:

$$\forall i \in [n], \qquad \left.\begin{array}{l} (\mathbf{r}, \underset{}{a}, \mathbf{s}) \in R \\ (\mathbf{r}, \underset{\substack{\uparrow \\ i}}{b}, \mathbf{s}) \in R \end{array}\right\} \implies a = b.$$

Rectangular critical relations played an important role in:

- Kearnes & Szendrei (2012), congruence modular case
- Zhuk (2017, 2020), "naked" special WNU case

Their study reduces to <u>completely functional</u> critical relations of arity ≥ 3.

Theorem

Suppose $\mathbf{A}_1, \ldots, \mathbf{A}_n$ are finite algebras in a locally finite Taylor variety, $n \geq 3$, and $R \leq_{sd} \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ is completely functional and critical.

1. Each $\mathbf{A}_i$ is subdirectly irreducible.
2. The monolith θ_i of $\mathbf{A}_i$ is abelian.
3. $\mathbf{D}(\theta_1) \cong \cdots \cong \mathbf{D}(\theta_n)$.
4. If $\mathbb{F}_i$ is the division ring associated to θ_i, then $\mathbb{F}_1 \cong \cdots \cong \mathbb{F}_n$.

Thus we may consider every θ_i-class of each $\mathbf{A}_i$ to be a vector space over a common finite field $\mathbb{F}$.

Let R^* be the unique upper cover of R in $\mathrm{Sub}(\mathbf{A}_1 \times \cdots \times \mathbf{A}_n)$.

5. Relative to R^*, R is "locally" the solution set of a single $\mathbb{F}$-linear equation.

For details, see slides from my BLAST 2025 Tutorial, lecture #3.

To learn more

1. R. Willard, Abelian congruences and similarity in varieties with a weak difference term, arXiv:2502.20517 (v3), 2026.
2. _______, Zhuk's bridges, centrality, and similarity, arXiv:2503.03551 (v2), 2026.
3. _______, BLAST 2025 tutorial slides, https://math.colorado.edu/blast/2025/schedule.html

Thank you!