Adjointable maps between Hermitian spaces and between orthosets

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Background

Foundational issues in quantum mechanics

The basic model of quantum mechanics is the

complex Hilbert space.

Can we reduce it to something simpler?

That is, can we extract from a Hilbert space H some algebraic or relational structure that is easier to understand and allows us to reconstruct H?

Approaches to structural reduction

The logico-algebraic approach

Excluding small dimensions, we have a correspondence:

$$\text{complex Hilbert spaces} \quad \rightleftarrows \quad \text{certain ortholattices}$$

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Issue in both cases

What about the respective structure-preserving maps?

Linear and projective spaces

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Definition

An (irreducible) projective space (with 0) is a set P equipped with an operation $\star\colon P \times P \to \mathcal{P}(P)$ and a constant 0, such that:

(P1) For any $a, b \in P$, $\{0, a, b\} \subseteq a \star b$ and $a \star b$ contains a further element if and only if $0, a, b$ are pairwise distinct.

(P2) For distinct non-zero elements $c, d \in a \star b$, we have $a \star b = c \star d$.

(P3) For pairwise distinct non-zero elements a, b, c, d, we have $a \star b \ \cap \ c \star d = \{0\}$ if and only if $a \star c \ \cap \ b \star d = \{0\}$.

Linear and projective spaces

A first step towards structural reduction of a linear space is to switch to the projective space.

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(P2) For distinct non-zero elements $c, d \in a \star b$, we have $a \star b = c \star d$.

(P3) For pairwise distinct non-zero elements a, b, c, d, we have $a \star b \ \cap \ c \star d = \{0\}$ if and only if $a \star c \ \cap \ b \star d = \{0\}$.

We call $a \star b$ the line spanned by a and b,
and we call elements $a \neq 0$ proper.

The correspondence of objects

Coordinatisation Theorem of Projective Geometry

Let V be a linear space and let

$$P(V) \;=\; \{\langle u\rangle : u \in V\}.$$

For $u, v \in V$, we put

$$\langle u\rangle \star \langle v\rangle \;=\; \{\langle w\rangle : w \in \langle u, v\rangle\}.$$

Then $P(V)$, equipped with $\star$ and the zero subspace $\{0\}$, is a projective space.

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Then $P(V)$, equipped with $\star$ and the zero subspace $\{0\}$, is a projective space.

Conversely, let P be a projective space of rank $\geqslant 4$. Then there is a linear space V such that $P(V)$ is isomorphic to P.

Structure-preserving maps

Definition

Let V_1 and V_2 be linear spaces over sfields F_1 and F_2.

We call $\varphi\colon V_1 \to V_2$ semilinear if

$$\varphi(u+v) \;=\; \varphi(u)+\varphi(v), \quad u,v \in V_1$$

and there is a homomorphism $\sigma\colon F_1 \to F_2$ such that

$$\varphi(\alpha u) \;=\; \alpha^{\sigma}\varphi(u), \quad u \in V_1,\;\; \alpha \in F_1.$$

If σ is an isomorphism, we call φ quasilinear.

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Definition

Let P_1 and P_2 be projective spaces.

We call $f\colon P_1 \to P_2$ a projective homomorphism if, for $a,b \in P_1$,

$$a \in b \star c \text{ implies } f(a) \in f(b) \star f(c),$$
$$f(0) = 0.$$

The correspondence of maps

Fundamental Theorem of Projective Geometry
(Faure, Frölicher)

Let $\varphi\colon V_1 \to V_2$ be a semilinear map. Then

$$P(\varphi)\colon P(V_1) \to P(V_2),\ \langle u \rangle \mapsto \langle \varphi(u) \rangle$$

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is a projective homomorphism.

Conversely, let $f\colon P(V_1) \to P(V_2)$ be a projective homomorphism whose image is not contained in a line.

Then there is a semilinear map $\varphi\colon V_1 \to V_2$ such that $f = P(\varphi)$.

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A Hermitian space is a linear space H over an $\star$-sfield F together with a product $(\cdot\,|\,\cdot) : H \times H \to F$ such that, for any $u, v, w \in H$ and $\alpha, \beta \in F$:

$$
\begin{aligned}
(\alpha u + \beta v \,|\, w) &= \alpha\ (u \,|\, w) + \beta\ (v \,|\, w)\,, \\
(w \,|\, \alpha u + \beta v) &= (w \,|\, u)\ \alpha^\star + (w \,|\, v)\ \beta^\star, \\
(u \,|\, v) &= (v \,|\, u)^\star\,, \\
(u \,|\, u) = 0 &\text{ implies } u = 0.
\end{aligned}
$$

Orthosets

Definition

An orthoset is a set X equipped with a binary relation $\perp$ and a constant 0 such that:

- $a \perp b$ implies $b \perp a$,
- $a \perp a$ only if $a = 0$,
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The supremum of the cardinalities of sets of mutually orthogonal elements is the rank of X.

Example

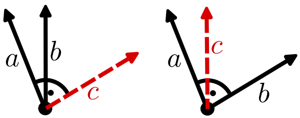
Let H be a Hermitian space. Then $P(H)$, equipped with the usual orthogonality relation $\perp$ and with $\langle 0 \rangle$, is an orthoset.

The correspondence of objects

Definition

An orthoset X is called linear if, for any distinct proper elements a and b, there is a proper element c such that

- exactly one of b and c is orthogonal to a and
- $\{a, b\}^{\perp} = \{a, c\}^{\perp}$.



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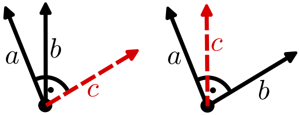

Coordinatisation Theorem for Linear Orthosets
(J. Paseka, Th.V.)

Let H be a Hermitian space. Then $P(H)$, together with $\perp$ and $\langle 0 \rangle$, is a linear orthoset.

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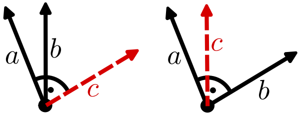

Coordinatisation Theorem for Linear Orthosets (J. Paseka, Th.V.)

Let H be a Hermitian space. Then $P(H)$, together with $\perp$ and $\langle 0 \rangle$, is a linear orthoset.

Conversely, let X be a linear orthoset of rank $\geqslant 4$. Then there is a Hermitian space H such that $P(H)$ is isomorphic to X.

Structure-preserving maps

Definition

We call a linear map $\varphi\colon H_1 \to H_2$ between Hermitian spaces adjointable if there is a linear map $\varphi^*\colon H_2 \to H_1$ such that

$$(\varphi(u) \,|\, v) \;=\; (u \,|\, \varphi^*(v)) \quad \text{for any } u \in H_1 \text{ and } v \in H_2.$$

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Example

A linear map between Hilbert spaces is adjointable iff it is continuous.

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Example

A linear map between Hilbert spaces is adjointable iff it is continuous.

Definition

We call a map $f\colon X \to Y$ between orthosets adjointable if there is a map $g\colon Y \to X$ such that, for any $x \in X$ and $y \in Y$,

$$f(x) \perp y \text{ if and only if } x \perp g(y).$$

The correspondence of maps: the easy direction

Proposition

Let $\varphi\colon H_1 \to H_2$ be an adjointable linear map between Hermitian spaces.

Then $P(\varphi)$ is adjointable and its adjoint is $P(\varphi)^* = P(\varphi^*)$.

The correspondence of maps: the easy direction

Proposition

Let $\varphi\colon H_1 \to H_2$ be an adjointable linear map between Hermitian spaces.

Then $P(\varphi)$ is adjointable and its adjoint is $P(\varphi)^* = P(\varphi^*)$.

Proof. For any $u \in H_1$ and $v \in H_2$, we have

$$
\begin{aligned}
&P(\varphi)(\langle u\rangle) \perp \langle v\rangle \\
\text{iff } &\varphi(u) \perp v \\
\text{iff } &(\varphi(u) \mid v) = 0 \\
\text{iff } &(u \mid \varphi^*(v)) = 0 \\
\text{iff } &u \perp \varphi^*(v) \\
\text{iff } &\langle u\rangle \perp P(\varphi^*)(\langle v\rangle).
\end{aligned}
$$

Consequences of adjointability

Lemma

Let $f\colon X \to Y$ be an adjointable map between orthosets. Then

- for any $A \subseteq X$, we have $f(A^{\perp\perp}) \subseteq f(A)^{\perp\perp}$, and
- $f(0) = 0$.

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Let $f\colon X \to Y$ be an adjointable map between orthosets. Then

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In particular, we then have

$$f(\{a, b\}^{\perp\perp}) \subseteq \{f(a), f(b)\}^{\perp\perp}$$

for any $a, b \in X$, hence:

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In particular, we then have

$$f(\{a, b\}^{\perp\perp}) \subseteq \{f(a), f(b)\}^{\perp\perp}$$

for any $a, b \in X$, hence:

Corollary

Any adjointable map $f\colon P(H_1) \to P(H_2)$ between orthosets is a projective homomorphism.

Lemma

Let $\varphi\colon H_1 \to H_2$ and $\psi\colon H_2 \to H_1$ be semilinear maps such that $P(\varphi)$ and $P(\psi)$ form an adjoint pair.
Assume furthermore that φ spans a subspace of dimension $\geqslant 2$.
Then φ is quasilinear.

The correspondence of maps: the non-trivial direction

Lemma

Let $\varphi\colon H_1 \to H_2$ and $\psi\colon H_2 \to H_1$ be semilinear maps such that $P(\varphi)$ and $P(\psi)$ form an adjoint pair.
Assume furthermore that φ spans a subspace of dimension $\geqslant 2$.
Then φ is quasilinear.

Theorem (J. Paseka, Th.V.)

Let H_1 and H_2 be Hermitian spaces.
Let $f\colon P(H_1) \to P(H_2)$ be an adjointable map of rank $\geqslant 3$.

Then there is a quasilinear map $\varphi\colon H_1 \to H_2$ such that $f = P(\varphi)$.

Adjointability of the inducing map

Theorem (J. Paseka, Th.V.)

Let $\varphi\colon H_1 \to H_2$ be a linear map between Hermitian spaces.

Then φ is adjointable if and only if $P(\varphi)$ is adjointable.

Adjointability of the inducing map

Theorem (J. Paseka, Th.V.)

Let $\varphi\colon H_1 \to H_2$ be a linear map between Hermitian spaces.

Then φ is adjointable if and only if $P(\varphi)$ is adjointable.

Corollary

For a linear map $\varphi\colon H_1 \to H_2$ between Hilbert spaces, the following are equivalent:

- φ is continuous;
- φ is bounded;
- φ is adjointable;
- $P(\varphi)$ is adjointable.

Preservation of orthogonality

Definition

An orthoisomorphism is a bijection $f\colon X \to Y$ between orthosets such that

$$x \perp y \text{ if and only if } f(x) \perp f(y)$$

for any $x, y \in X$.

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Proposition

Let $f\colon X \to Y$ be a bijection between orthosets. Then f is an orthoisomorphism if and only if f and f^{-1} form an adjoint pair.

Piziak's Theorem

Theorem (R. Piziak; J. Paseka, Th.V.)

Let H_1 be an at least 2-dimensional linear space over F_1, equipped with a non-degenerate sesquilinear form $(\cdot\,|\,\cdot)_1$, and let H_2 be a linear space over F_2 equipped with a sesquilinear form $(\cdot\,|\,\cdot)_2$.

Let $\varphi\colon H_1 \to H_2$ be a semilinear map with associated homomorphism σ and assume that

$$u \perp v \text{ implies } \varphi(u) \perp \varphi(v)$$

for any $u, v \in H_1$.

Then there is a unique $\lambda \in F_2$ such that

$$(\varphi(u)\,|\,\varphi(v))_2 \;=\; (u\,|\,v)_1{}^{\sigma}\,\lambda$$

for any $u, v \in H_1$.

Definition

We call a bijection $\varphi\colon H_1 \to H_2$ between Hermitian spaces quasiunitary if

- φ is quasilinear with associated isomorphism σ and
- there is a $\lambda \in F_2 \setminus \{0\}$ such that

$$(\varphi(u) \,|\, \varphi(v))_2 \;=\; (u \,|\, v)_1{}^{\sigma}\, \lambda$$

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Wigner's Theorem for Hermitian spaces

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for any $u, v \in H_1$.

Theorem (J. Paseka, Th.V.)

Let H_1 and H_2 be at least 3-dimensional Hermitian spaces, and let $f\colon P(H_1) \to P(H_2)$ be an orthoisomorphism.
Then there is a quasiunitary map $\varphi\colon H_1 \to H_2$ such that $f = P(\varphi)$.

Conclusion

Let H_1, H_2 be Hermitian spaces and
let $P(H_1), P(H_2)$ be the associated orthosets.

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The relationship between the structure-preserving maps $H_1 \to H_2$ and $P(H_1) \to P(H_2)$ is not perfect but we may notice:

- A map $f\colon P(H_1) \to P(H_2)$ is adjointable if there is another map $g\colon P(H_2) \to P(H_1)$ such that

$$f(\langle u \rangle) \perp \langle v \rangle \text{ iff } \langle u \rangle \perp g(\langle v \rangle)$$

  for any $u \in H_1$ and $v \in H_2$. Provided the rank is $\geqslant 3$, f is induced by a quasilinear map from H_1 to H_2.

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- A map $f\colon P(H_1) \to P(H_2)$ is an orthoismorphism if

$$\langle u \rangle \perp \langle v \rangle \text{ iff } f(\langle u \rangle) \perp f(\langle v \rangle).$$

  Provided that dimensions are $\geqslant 3$, f is induced by a quasiunitary map from H_1 to H_2.