Transitivity of the relation of being an ideal in a nearring

AAA 108
TU, Vienna

Presented by
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Introduction

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For example:

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- For a given ring N, for all rings J, I and A,

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  $J \lhd I \lhd A$ with $I/J \cong N$ implies $J \lhd A$
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A ring N fulfills condition (F) (or N is an F-ring) if:
$J \lhd I \lhd A$ with $I/J \cong N$ implies $J \lhd A$.

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If $a0 = 0$ for all $a \in N$, then N is called *zero-symmetric*.
In the sequel, all nearrings are zero-symmetric.

Condition (F) for nearrings

For rings: The following four conditions are equivalent:

(1) R is an F-ring.

(2) R has middle annihilator zero

that is, $M_R := \{x \in R \mid RxR = 0\} = 0$.

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(4) R is *quasi-semiprime:* for $x, y \in R$,
$xr = yr$ for all $r \in R \Rightarrow x = y$ and
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There are F-nearrings that are not quasi-semiprime.

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Theorem

If N is an F-nearring, then
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Why is this a problem?

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Thus

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Multiplication with $i \in I$ on the left is no good
(no left distributivity).

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Kaarli (1992) defined a K-*nearring* N (or N has *property* (K)):

- For any N-subgroup X of $N \oplus N$ with $nx = ny$ for all $\forall (x, y) \in X, \forall n \in N$, necessarily $x = y$.

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For rings it is: N is a K-ring $\quad \Leftrightarrow \quad$ N is an F-ring.

Right and middle annihilators for nearrings

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There are nearrings N with $R_N = 0 = (0 : N)_N$, but N is not an F-nearring.

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Trivial nearrings.

Let $(T, +)$ be any group, not necessarily commutative, and let S be a subset of T with $0 \in S$.

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If $S = T$, then $ab = 0$ for all $a, b \in T$;
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Theorem

The following four conditions are equivalent for the nearring T:
(1) T is an F-nearring.
(2) T is a K-nearring.
(3) T is 2-primitive.
(4) The subset S of T contains no nonzero subgroups of T.

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When $S = \{0\}$, then $N = M_0(G)$ which is a nearring with identitity; hence a K-nearring and of no further interest.
Suppose thus S has at least two distinct elements.

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If $S = G \setminus \{a\}$ for some fixed $0 \neq a \in G$, then N is a trivial nearring in the sense of the previous example and thus also of no further interest.

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$\Rightarrow$ N is an F-nearring.

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