Grouplike algebras characterized in lattice language

Branimir Šešelja
University of Novi Sad
Joint research with
Andreja Tepavčević and **Jelena Jovanović**

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To our friend Günther

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The most common algebras we analyze are those with binary operations, e.g., semigroups, loops, rings, some classes of lattices, boolean algebras. We also deal with other algebras, in particular those in coherent varieties, which consequently have regular congruences.

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Let $n, b \in \downarrow a$, $n \leqslant b$. We say that n is **normal in** $\downarrow b$, we denote it by $n \blacktriangleleft b$, if $n = x_a$, for some $x \in [b, \overline{b}]$. Equivalently,
$n \blacktriangleleft b$ if and only if $[\overline{n}, \overline{n} \vee b] \cap T_a = \{\overline{n}\}$.

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Figure 1

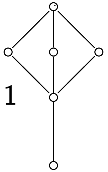

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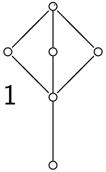
Figure 1  Q

We say that L an ***A*-lattice** if it is a modular lattice with normal elements determined by a in which $\downarrow a$ does not have an interval-sublattice which is isomorphic with the lattice Q in Fig. 1; Q represents the subgroup lattice of the quaternion group, which is uniquely determined by its subgroup lattice.

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Therefore, under the map $m_\Delta : \mathrm{Wcon}(\mathcal{A}) \longrightarrow \downarrow\Delta$ given by $m_\Delta(x) = \Delta \wedge x$, the lattice $\mathrm{Sub}(\mathcal{A})$ of subalgebras of $\mathcal{A}$ is up to the isomorphism a retract of the lattice $\mathrm{Wcon}(\mathcal{A})$ of weak congruences of this algebra.

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The classes of the kernel φ_Δ of m_Δ are congruence lattices of subalgebras of $\mathcal{A}$:
If $x \in {\downarrow}\Delta$, then $x = \Delta_B$ for a subalgebra $\mathcal{B}$ of $\mathcal{A}$, and the top element $\overline{[\Delta_B]_{\varphi_\Delta}}$ of the class $[x]_{\varphi_\Delta}$ is the square B^2. Therefore,

$$[\Delta_B]_{\varphi_\Delta} = [\Delta_B, B^2] \cong \mathrm{Con}(\mathcal{B}).$$

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It turned out that main properties of groups, related to class operators S, H and P, can be equivalently formulated or represented in lattice terms.
Also the other structural and algebraic properties like series and systems of subgroups, commutator subgroups, center and related notions, have their lattice-theoretic definitions and interpretations.

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Our first results related to weak congruence lattices of groups:

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- G. Czédli, B. Šešelja, A. Tepavčević, *Semidistributive elements in lattices; application to groups and rings*, Algebra Univers. 58 (2008) 349–355.
- G. Czédli, M. Erné, B. Šešelja, A. Tepavčević, *Characteristic triangles of closure operators with applications in general algebra*, Algebra Univers. 62 (2009) 399–418.

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Then we continued:

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- J. Jovanović, B. Šešelja, A. Tepavčević, *Lattice characterization of finite nilpotent groups*, Algebra Univers. 2021 82(3) 1–14.
- J. Jovanović, B. Šešelja, A. Tepavčević, *Lattices with normal elements* Algebra Univers. 2022 83(1) 1–28.
- M.Z. Grulović, J. Jovanović, B. Šešelja, A. Tepavčević, *Lattice Characterization Of Some Classes Of Groups By Series Of Subgroups*, International Journal of Algebra and Computation, 2023 33(02) 211–235.

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- J. Jovanović, B. Šešelja, A. Tepavčević, *Nilpotent groups in lattice framework*, Algebra univers. 2024 85(4) 40.
- M.Z. Grulović, J. Jovanović, B. Šešelja, A. Tepavčević, *On classes of groups characterized by classes of lattices*, Czech Math J 75, 1213–1228 (2025).
- M.Z. Grulović, J. Jovanović, B. Šešelja, A. Tepavčević, *Systems of subgroups and Kurosh-Chernikov classes of groups in lattice framework* (submitted).

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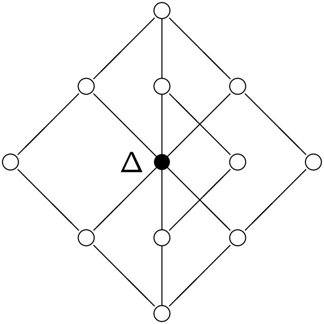
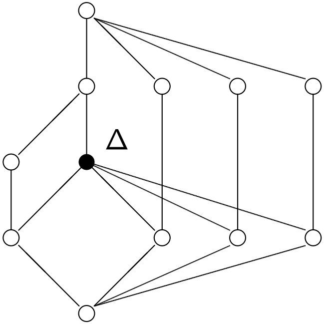
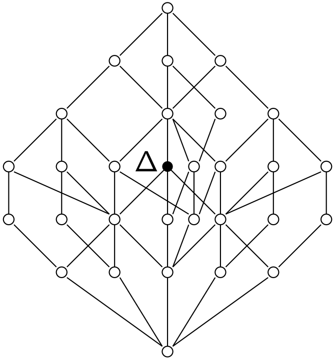


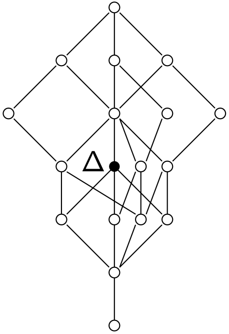
Wcon($\mathcal{K}$)



Wcon($\mathcal{S}_3$)



dihedral group of order 8



quaternion group

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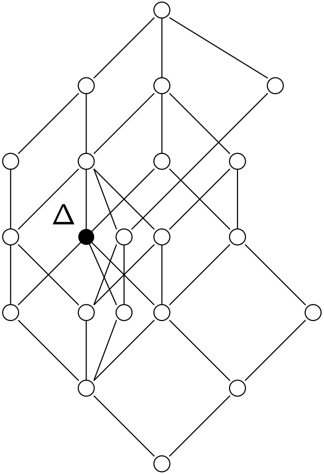

Wcon(R)

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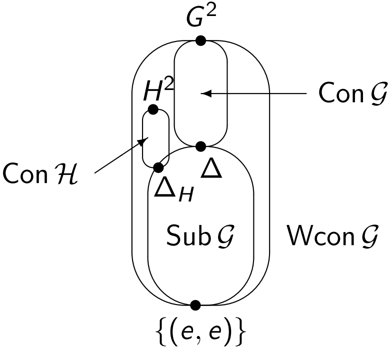

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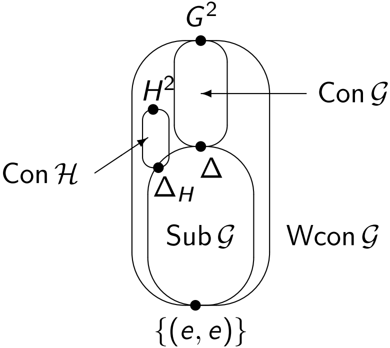
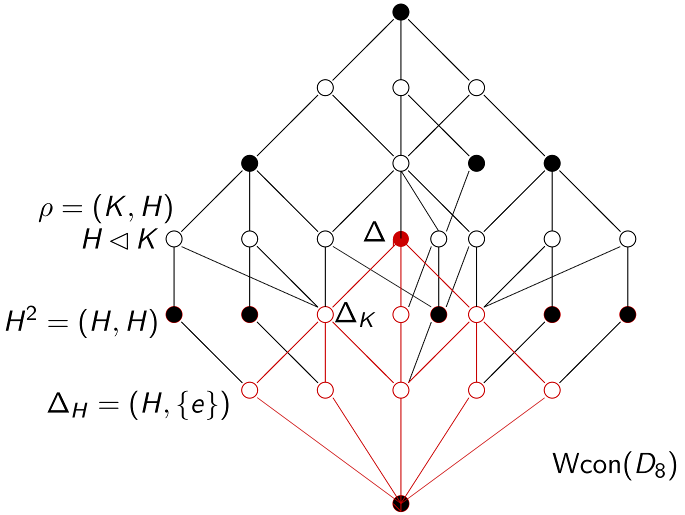


Therefore, for a group G, the lattice $\mathrm{Wcon}(G)$ can be represented as a collection of all ordered pairs (H, K) of subgroups of G, so that $K \lhd H$.


$\rho = (K, H)$
$H \lhd K$
Δ
Δ_K
$H^2 = (H, H)$
$\Delta_H = (H, \{e\})$
$\mathrm{Wcon}(D_8)$

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$$[0,\overline{c_1}],[\overline{c_1},\overline{c_2}],\dots,[\overline{c_k},1]$$

so that for every $i \in \{0,1,\dots,k\}$ the following holds:

(a) $c_i \blacktriangleleft a$;

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Theorem

The C-class of groups consists of all nilpotent groups.

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We also say that the above defined structures are the **grouplike algebras**.

Proposition

Let $\mathcal{A}$ *be an algebra in the variety* $\mathcal{Q}$ *and* $\theta \in \mathrm{Wcon}(\mathcal{A})$, *more precisely let* $\theta \in [\Delta_C, C^2]$, *where* $\mathcal{C}$ *is a subalgebra of* $\mathcal{A}$. *Then* $\theta = B^2 \vee \Delta_C$, *for some subalgebra* $\mathcal{B}$ *of* $\mathcal{A}$, $B \leqslant C$.

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Let $\mathcal{B}, \mathcal{C}$ be subalgebras od $\mathcal{A}$, all belonging to the variety $\mathcal{Q}$. We say that $\mathcal{B}$ is **normal** in $\mathcal{C}$, if for the diagonal elements Δ_B and Δ_C, in $\mathrm{Wcon}(\mathcal{A})$, we have

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Indeed, as proved by Geiger, *a coherent variety is congruence regular.*

We analyze the structure of the weak congruence lattice for algebras in the variety of the type $\mathcal{Q}$ (grouplike algebras).

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Consequently, for a congruence θ on $\mathcal{A}$, the lattice Wcon$(\mathcal{A}/\theta)$ *is the set union of intervals $[\theta \wedge B^2, B^2]$, for every subalgebra $\mathcal{B}$ of $\mathcal{A}$ which is the union of classes of θ.*

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Proposition

Let $\mathcal{A}$ be an algebra in the variety $\mathcal{Q}$ and $\theta \in \mathrm{Wcon}(\mathcal{A})$, i.e., let $\theta \in [\Delta_C, C^2]$, where $\mathcal{C}$ is a subalgebra of $\mathcal{A}$. Then $\theta = B^2 \vee \Delta_C$, for some subalgebra $\mathcal{B}$ of $\mathcal{A}$, $B \leqslant C$.

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Theorem

Let $\mathcal{A}$ be an algebra from the variety $\mathcal{Q}$ and let $\mathcal{B} \leqslant \mathcal{C}$, $\mathcal{B}, \mathcal{C} \in \mathrm{Sub}(\mathcal{A})$. Then $\Delta_B \blacktriangleleft \Delta_C$ in the lattice $\mathrm{Wcon}(\mathcal{A})$ if and only if there is $\theta \in [\Delta_C, C^2]$, so that $\theta = B^2 \vee \Delta_C$.

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Corollary

Under the above conditions, $\mathrm{Wcon}(\mathcal{C}/\theta) \cong [B^2, C^2]$ where the main diagonal of this lattice corresponds to θ.

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$+$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$*$	e	a	b	c
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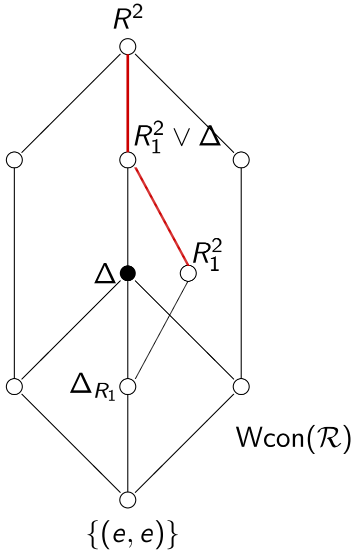
The weak congruence lattice is presented by the diagram.


R^2
$R_1^2 \vee \Delta$
R_1^2
Δ
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$\mathrm{Wcon}(\mathcal{R})$
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We have that $\Delta_{R_1} \blacktriangleleft \Delta$;
In the diagram of Wcon($\mathcal{R}$) we can identify the interval $[R_1^2, R^2]$, which is, as a lattice, isomorphic to Wcon($\mathcal{R}/\theta$), where $\theta = R^2 \vee \Delta$, i.e., θ is the congruence on $\mathcal{R}$, corresponding to the ideal $\mathcal{R}_1$.

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(iii) R has the CEP and the CIP.
(iv) for any two ideals I_1 and I_2 on arbitrary subrings we have $\overline{I_1} \cap \overline{I_1} = \overline{I_1 \cap I_2}$, where $\overline{I_j}$ is the smallest ideal on the ring collapsing the ideal I_j on a subring.

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Following particular classes of groups we say that an algebra $\mathcal{A}$ in the variety $\mathcal{Q}$ is a ***T*-algebra** if $\mathcal{B} \lhd \mathcal{C} \lhd \mathcal{A}$ implies $\mathcal{B} \lhd \mathcal{A}$.

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$\mathcal{A}$ is a **T^*-algebra**, if for every subalgebra $\mathcal{D}$ of $\mathcal{A}$, $\mathcal{B} \lhd \mathcal{C} \lhd \mathcal{D} \leqslant \mathcal{A}$ implies $\mathcal{B} \lhd \mathcal{D}$.

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Clearly, *$\mathcal{A}$ is a T^*-algebra if and only if every subalgebra of $\mathcal{A}$ is a T-algebra. The class of T^*-algebras consists of all algebrass in which normality is a transitive relation.*

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We say that a lattice L with normal elements determined by a has the **T-chain property** if for every $x \in \uparrow a$, $[x_a, \overline{x_a}] \cong [a, x]$ with respect to the map $k_x : y \mapsto y \vee a$.

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More generally, a lattice L with normal elements determined by a has a T^*-**chain property** if for every $x \in L$, $[x_a, \overline{x_a}] \cong [x \wedge a, x]$ under $h_x : y \mapsto y \vee (x \wedge a)$.

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Let L be a lattice with normal elements determined by a.
(i) *For all $b, c \in \downarrow a$, $b \blacktriangleleft c \blacktriangleleft a$ implies $b \blacktriangleleft a$, if and only if L has the T-chain property.*
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Proposition

If L is a lattice L with normal elements determined by a and a is modular and cancellable, then the normality relation $\blacktriangleleft$ is transitive in L.

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Theorem

An algebra $\mathcal{A}$ is a T-algebra (T^*-algebra) if and only if the weak congruence lattice $\mathrm{Wcon}(\mathcal{A})$ *of $\mathcal{A}$ has the T-chain property (T^*-chain property).*

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Theorem

A ring R has the SNC if and only if the lattice $\mathrm{Wcon}(R)$ fulfills the following: for every $\Delta_S \in \downarrow\Delta$, $\Delta_S \neq \Delta$, there is Δ_T such that $\Delta_S \blacktriangleleft \Delta_T$. A ring R satisfies the WNC if and only if in the lattice $\mathrm{Wcon}(R)$ for every coatom Δ_S of the sublattice $\downarrow\Delta$ we have $\Delta_S \blacktriangleleft \Delta$.

An endomorphism φ of an algebra $\mathcal{A}$ is said to be **strong** if φ is compatible with all congruences on $\mathcal{A}$. If every congruence on $\mathcal{A}$ is a kernel of some strong endomorphism of $\mathcal{A}$, then $\mathcal{A}$ is said to have a **strong endomorphism kernel property (SEKP)**.

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Theorem (Ghumashyan, Gurican)

If $\mathcal{A}$ is a group, then an endomorphism φ on $\mathcal{A}$ is strong if and only if for every normal subgroup N, $\varphi(N) \subseteq N$.

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Theorem

Let $\mathcal{A}$ be an algebra in the variety $\mathcal{Q}$ which fulfills the Strong Endomorphism Kernel Property (the SEKP) and let $\mathcal{B}$ be a subalgebra of $\mathcal{A}$, fulfilling $\Delta_B \blacktriangleleft \Delta$. Let also θ be a congruence on $\mathcal{A}$, such that $\theta = B^2 \vee \Delta$ in the lattice $\mathrm{Wcon}(\mathcal{A})$.

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Then, the filter $\uparrow B^2$ as a weak congruence lattice of $\mathcal{A}/\theta$ with the main diagonal $B^2 \vee \Delta$ is isomorphic to the sublattice of the ideal $\downarrow B^2$, which represents the weak congruence lattice of $h(\mathcal{A})$, where h is the endomorphism whose kernel is θ.

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A property P is said to be **residual in a class of algebras** if every algebra of this class that residually has the property P also has the property P itself.

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To investigate L-classes of algebras we use the following.

Proposition

An algebra $\mathcal{A}$ is a semidirect product of its subalgebras $\mathcal{B}$ and $\mathcal{C}$ if and only if the following holds in the lattice Wcon$\mathcal{A}$)*:*
$\Delta_{\mathcal{B}} \blacktriangleleft \Delta$; $\Delta_{\mathcal{B}} \vee \Delta_{\mathcal{C}} = \Delta$ *and* $\Delta_{\mathcal{B}} \wedge \Delta_{\mathcal{C}} = \{(e, e)\}$.

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Proposition

Let $\mathcal{A}$ be an algebra in a class $\mathfrak{P}$. If $\Delta_{\mathcal{B}} \blacktriangleleft \Delta$ in the lattice Wcon($\mathcal{A}$)*, then $\mathcal{A}$ is the extension of the subalgebra $\mathcal{B}$ (by the algebra $\mathcal{A}/\mathcal{B}$).*

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Proposition

If $\mathfrak{P}$ and $\mathfrak{Q}$ are L-classes, then an algebra $\mathcal{A}$ is a $\mathfrak{P}$-by-$\mathfrak{Q}$ algebra (it belongs to the class $\mathfrak{P}\mathfrak{Q}$) if and only if in the lattice $\mathrm{Wcon}(\mathcal{A})$ there exists some $\Delta_B \blacktriangleleft \Delta$ such that sublattices $\downarrow B^2$ and $\uparrow B^2$ fulfil lattice theoretic properties $L_{\mathfrak{P}}$ and $L_{\mathfrak{Q}}$, respectively.

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Proposition

Let $\mathfrak{P}$ be an L-class of algebras. An algebra $\mathcal{A}$ is a residually $\mathfrak{P}$-algebra (it belongs to the class $\mathbf{R}\mathfrak{P}$) if and only if the lattice $\mathrm{Wcon}(\mathcal{A})$ *fulfils:*
(∗) *For each nontrivial $\Delta_X \in \mathrm{C}(\downarrow\Delta)$, there is $\Delta_B \blacktriangleleft \Delta$, such that $\Delta_B \wedge \Delta_X < \Delta_X$ and the interval $[N^2, A^2]$, as the lattice with normal elements determined by $B^2 \vee \Delta$, satisfies the lattice theoretic properties $L_{\mathfrak{P}}$.*

Corollary

If $\mathfrak{P}$ is an L-class, so is the class of residually $\mathfrak{P}$-algebras $\mathbf{R}\mathfrak{P}$.

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Proposition

An L*-class $\mathfrak{P}$ of algebras is* S*-closed (*S_n*-closed) if and only if for every algebra $\mathcal{A}$ in $\mathfrak{P}$, the following holds:*
For every $\Delta_B \in {\downarrow}\Delta$ $\Delta_B \blacktriangleleft \Delta$, the ideal ${\downarrow}B^2$ as a lattice with normal elements determined by Δ_B, fulfils $\mathsf{L}_{\mathfrak{P}}$, i.e., the lattice theoretic properties defining the class $\mathfrak{P}$ as an L*-class.*

Corollary

If $\mathfrak{P}$ is an L-class, so is the class of residually $\mathfrak{P}$-algebras $\mathbf{R}\mathfrak{P}$.

Proposition

An L*-class* $\mathfrak{P}$ *of algebras is* S*-closed (*S_n*-closed) if and only if for every algebra* $\mathcal{A}$ *in* $\mathfrak{P}$*, the following holds:*
For every $\Delta_B \in \downarrow\Delta$ $\Delta_B \blacktriangleleft \Delta$*, the ideal* $\downarrow B^2$ *as a lattice with normal elements determined by* Δ_B*, fulfils* $L_{\mathfrak{P}}$*, i.e., the lattice theoretic properties defining the class* $\mathfrak{P}$ *as an* L*-class.*

Proposition

An L*-class* $\mathfrak{P}$ *of algebras is* H*-closed if and only if for every algebra* $\mathcal{A}$ *in* $\mathfrak{P}$*, the following holds:*
For every $\Delta_B \in \downarrow\Delta$ *such that* $\Delta_B \blacktriangleleft \Delta$*, the interval* $[B^2, A^2]$ *as a lattice with normal elements determined by* $B^2 \vee \Delta$*, fulfils* $L_{\mathfrak{P}}$*, i.e., the lattice theoretic properties defining the class* $\mathfrak{P}$ *as an* L*-class.*

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Theorem

An L*-class* $\mathfrak{P}$ *of algebras is a variety if and only if the following hold:*
(i) if $\mathcal{A}$ *is a* $\mathfrak{P}$*-algebra, then in the lattice* $\mathrm{Wcon}(\mathcal{A})$ *for every* $\Delta_B \in \downarrow\Delta$*, such that* $\Delta_B \blacktriangleleft \Delta$*, the interval* $[B^2, A^2]$*, which is a lattice with normal element determined by* $B^2 \vee \Delta$*, satisfies* $L_{\mathfrak{P}}$*, i.e., the lattice properties determining the class* $\mathfrak{P}$ *and*
(ii) every algebra $\mathcal{A}$*, such that the lattice* $\mathrm{Wcon}(\mathcal{A})$ *satisfies* $(*)$*, belongs to* $\mathfrak{P}$*.*

Corollary

If a class of algebras $\mathfrak{P}$ is an L*-class with respect to a set of lattice identities $L_{\mathfrak{P}}$, then $\mathfrak{P}$ is a variety if and only if the weak congruence lattices of its members satisfy the property* $(*)$.

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Thanks for watching!