Strictly n-finite Varieties of Heyting algebras

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A **Heyting algebra** $(H, \wedge, \vee, \to, 0, 1)$ is a bounded distributive lattice with a binary operation $\to$ such that, for every $a, b, c \in H$:

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- Crucially: Heyting algebras are *not locally finite*.

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- **Quackenbush**: *a locally finite variety* V *with finitely many finite subdirectly irreducible members has no infinite subdirectly irreducible algebra*.
- **McKinsey and Valeriote**: *full characterisation of the locally finite varieties with a decidable equational theory*.

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In algebraic terms, the theorem above means that the *variety of Heyting algebras is not finitely-generated*.

Rieger-Nishimura Lattice

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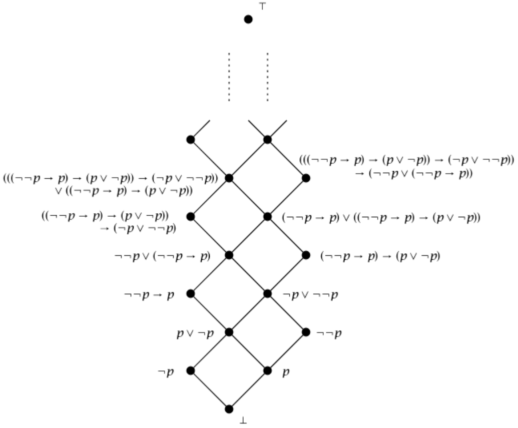

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Alternatively, what can we say about varieties between HA and BA?

Definition

An **intermediate logic** is a set of formulas L such that $\mathrm{IPC} \subseteq L \subseteq \mathrm{CPC}$, and such that L is closed under modus ponens and uniform substitution.

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However, in its general form, Maksimova's problem is **still open**.

Locally Finite Heyting Algebras

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Theorem (Bezhanishvili-Grigolia)

Let V *be a variety of Heyting algebras, then the following conditions are equivalent:*

(i) V *is locally finite.*

(ii) *The* V*-coproduct of any two finite V-algebras is finite.*

(iii) *Finite* V*-copowers of finite* V*-algebras are finite.*

(iv) *Either* $\mathsf{V} = \mathsf{BA}$ *or finite* V*-copowers of* $\mathbf{3} \in \mathsf{V}$ *are finite (where* $\mathbf{3}$ *refers to the three-element chain).*

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Problem (Bezhanishvili-Grigolia)

Let V *be a variety of Heyting algebras which is not locally finite, does it follow that the* V*-free algebra on two generators is infinite?*

Width 2 Case

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Theorem (Benjamins)

If V *is a variety of Heyting algebras with width 2, then it is locally finite if and only if its 2-generated subalgebras are finite.*

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For each $n < \omega$, there exists a variety of Heyting algebras V_n such that:

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(ii) V_n *contains an infinite $(n+1)$-generated algebra.*

Proof technique

- **Esakia duality**: every Heyting algebra is isomorphism to the clopen upset algebra of its dual Esakia space, the space of its prime filters ordered by reverse inclusion.

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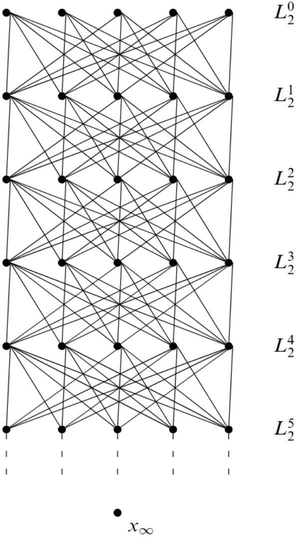
The Esakia space X_2



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