Factorizations of non-commutative polynomials

Phạm Ngọc Ánh

HUN-REN Rényi Institute

AAA 108 Vienna, February 6 – 8, 2026

Aims and tools

A **free algebra** is a free monoid or group algebra of rank ≥ 2. (Non-commutative) polynomials: elements of a free algebra.

Cohn [2]: Free ideal ring (fir) $\iff$ one-sided ideals are free of unique rank; its elements have irreducible factorizations. Free algebras are firs.

Aims: Find irreducible factorizations; uniqueness, algorithm?

Tools: link (=Sato) modules, lattices and localizations.

Cohn's example

$$xyzyx + xyz + zyx + xyx + x + z = (xyz + x + z)(yx + 1) =$$

=(xy+1)(zyx+x+z)

Notation

1. K a commutative ring, mainly a field or a pid.
2. The free monoid (non-commutative polynomial) algebra

$$A = K\langle x_1, \dots, x_n\rangle = K\langle t_1, \dots, t_n\rangle; t_i = x_i + 1$$

3. F_n the free group of rank n on t_i; $\Lambda = KF_n, R \in \{A, \Lambda\}$.
4. The non-commutative power series algebra

$$\Gamma = K\langle\langle x_1, \dots x_n\rangle\rangle; \epsilon\colon \Gamma \to K : x_i \mapsto 0.$$

R an augmented algebra via $\epsilon = \epsilon_{|_R} : R \to K$; the augmentation ideal $I = \ker \epsilon$ is free of rank n.

$$\Lambda = K\langle t_1, \dots, t_n; y_1, \dots, y_n\rangle / \langle t_i y_i - y_i t_i, 1 - t_i y_i\rangle \Longrightarrow$$

$$t_i^{-1} = y_i = 1 - x_i + x_i^2 - x_i^3 + \cdots + (-1)^l x_i^l + \cdots \in \Gamma.$$

$0 \neq f \in K[x] = A \Longrightarrow f = x^l \bar{f} \quad \& \quad \bar{f}(0) \neq 0$; l the *order* of f. A canonical bijection between irreducible factorizations of f and composition chains of coker $f = A/Af$. One can assume $f(0) \neq 0$.

Division algorithm for a polynomial $f = 1 + xf_x$

f defines an action on A by $x * 1 = -f_x$ and $x * x^l = x^{l-1}$ for $l \geq 1$. This action denoted by ∂ acts like either a generalized inverse or a generalized derivation. Division by f can be carried out by using $*$-action successively. This yields an algorithm to obtain the remainder and the quotient in finitely many steps.

Fox derivations and localizations

The free algebra $R \in \{\Lambda, A\}$ don't determine uniquely free generating sets. Normal forms depend on a choice of x_i.

$$f \in \Gamma \Longrightarrow f - f(0) = \sum_i x_i f_i$$

$f_i = \partial_i f$: i-th *partial generalized derivation* or *canonical Fox derivation of* f.

The Fox algebra $L(R)$ is the localization of R universally inverting a row $(x_1, \dots, x_n)$ whose entries generate freely $I = \ker \epsilon$, respectively.

Warning. The Fox algebra $L(\Lambda)$ of Λ is a combination of two localizations, namely, universally inverting a set $\{t_i = x_i + 1\}$ and a row $(x_1, \dots, x_n)$ of A.

The Fox algebra $L(R)$

R^*: the subalgebra of R generated by entries of the inverse column of $(x_1, \ldots, x_n)$. A^* is generated by $\partial_1, \ldots, \partial_n$ but $R^* = \Lambda^*$ is generated by $\partial_1, \ldots, \partial_n; t_1\partial_1, \ldots, t_n\partial_n$ by

$$f - f(0) = \sum_i x_i f_i = \sum_i (t_i - 1) f_i = -\sum_i (t_i^{-1} - 1) t_i f_i.$$

The Fox algebra is an appropriate algebraic structure for a study of generalized integrals, i.e., a multiplication by the x_i together with generalized derivations, i.e., a multiplication by the $\partial_i, t_i\partial_i$.

An obvious application. One can define Fox derivations on $K[x_1, \ldots, x_n]$ making it an injective hull of the field K.

The Fox algebra and polynomials

A polynomial $f \in R \in \{A, \Lambda\}$ is *comonic* if $\epsilon(f) = f(0) = 1$.

Theorem 1

If $f \in R$ is comonic, then the R-module $\operatorname{coker} f = R/Rf$ becomes naturally an $L(R)$-module .

More generally

Theorem 2

If $\phi = (f_{ij})$ is a square matrix of arbitrary size over R with $(f_{ij}(0)) = 1$, then $\operatorname{coker} \phi$ becomes naturally a module over the Fox algebra $L(R)$.

Link modules

$R \in \{A, \Lambda\}$; $\epsilon\colon R \to K$; $I = \ker \epsilon \implies$ a free resolution

$$0 \to {}^nR \xrightarrow{(x_1 \cdots x_n)} R \xrightarrow{\epsilon} K \to 0\colon (x_1 \cdots x_n) \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} = \sum_{i=1}^{n} x_i r_i$$

Definition 3

${}_RM$ *weak link (Sato) module* if $\mathrm{Tor}_*^R(K, M) = 0$, or equivalently, $\exists\, M^n \cong M$ such that

$$(1) \qquad (m_1, \cdots, m_n) \in M^n \longmapsto \sum_{i=1}^{n} x_i m_i = m \in M.$$

If ${}_RM$ is finitely presented, then M *Sato module*.

Link modules and lattices

$m \mapsto m_i$ defines Fox derivations on M.

Theorem 4

Let K be a field. Then link modules are precisely modules described in Theorem 2. Moreover, as modules over the Fox algebra $L(R)$ link modules are finitely presented of finite length.

Definition 5

A lattice of a link module M is a finitely generated K-submodule N which is an R^*-module and $M = RN$ holds .

Theorem 6

Every finitely presented link module of finite length has a smallest finite-dimensional lattice if K is a field.

$R \in \{A, \Lambda = KF_n\}$ has rank n; K a field.

Theorem 7

For any two arbitrary polynomials $\gamma, \lambda \in R$ there is a uniquely determined comonic polynomial $\delta \in R$, called, a greatest common divisor *of γ, λ by the property that δ is a generator of the left ideal of $L(R)$ generated by γ and λ, that is, $L(R)\gamma + L(R)\lambda = L(R)\delta$*

Theorem 8

Let $\pi \in R$ be a comonic polynomial which is not a unit. Then π is irreducible iff the link module $\text{coker}\pi = R/R\pi$ is a simple $L(R)$-module iff the smallest lattice of $\text{coker}\,\pi$ is a finite-dimensional simple R^*-module.

The main result

Theorem 9

$\alpha \in R$ non-unital comonic; N the smallest finite-dimensional R^*-lattice of coker $\alpha = M$. *Any rrreducible factorization $\alpha = \rho_1 \cdot \rho_l$ corresponds to a composition chain of R^*-module N and $L(R)$-module M, respectively. Namely,*

$$0 \subseteq \frac{R\rho_2 \cdots \rho_l}{R\alpha} \subseteq \frac{R\rho_3 \cdots \rho_l}{R\alpha} \subseteq \cdots \subseteq \frac{R\rho_l}{R\alpha} \subseteq \frac{R}{R\alpha}$$

is a composition chain of M. The simple subfactor $\frac{R\rho_{j-1}\cdots\rho_l}{R\rho_j\cdots\rho_l}$ or coker ρ_l *determines ρ_j or ρ_l, respectively, only up to the similarity.*

$a, b \in R$ similar $\iff$ left (right) modules $R/Ra, R/Rb\,(R/aR, R/bR)$ isomorphic.

The idea and some difficulties of the proof

A has a degree function ensuring irreducible factorizations.
Λ has only an order function no degree function.
An order function is not suitable to studying factorizations.
Λ has a length function suitable to studying factorizations.
A length function reduces the process by quite slowly.

Basic references

- G. Abrams, P. Ara, M. Siles Molina, *Leavitt path algebras*, Lecture Notes in Mathematics **2191**, Springer 2018.
- P. N. Ánh, Link modules, localizations and factorizations, arXiv: 2510.03429
- P. N. Ánh, F. Mantese, Non-commutative factorizations and finite-dimensional representations of free algebras, arXiv:2403.1783
- P.N. Ánh, M. Siddoway, Module types of localizations, with applications to leavitt path algebras, *Israel J. Math.* (2025, 1 – 35; DOI: 10.1007/s11856-025-2819-x
- P. M. Cohn, *An Introduction to Ring Theory*, Springer 2000.

Basic references

W. G. Leavitt, The module type of a ring, *Trans. Amer. Math. Soc.* **103**(1)(1962), 113 – 130.

A. Rosenmann, S. Rosset, Ideals of finite codimension in free algebras and the fc-localization, *Pacific J. Math.* **162**(2)(1994), 351 - 371.

Thank you for your attention!