Quantales Carrying Ortholattice Structure

Michal Botur[1], David Kruml[2], Jan Paseka[2]

[1]Palacký University Olomouc, Czech Republic
[2]Masaryk University Brno, Czech Republic

e-mail: paseka@math.muni.cz

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Outline

Two Traditions in Non-Classical Logic

Many-Valued Logic

- Fuzzy logic
- Intuitionistic logic
- Linear logic
- Residuated structures
- Girard quantales

Quantum Logic

- Quantum mechanics
- Non-Boolean reasoning
- Incompatible observables
- Orthomodular lattices
- Hilbert space subspaces

Common Ground

Both traditions meet at Boolean algebras

Central Question

Can we find richer non-Boolean structures that satisfy both residuation principles and orthocomplementation?

Our Main Results

1. **Impossibility result**: Any complemented lattice with integral residuated structure must be Boolean (Theorem 2)
2. **Positive answer**: There exist orthomodular lattices that are also commutative Girard quantales but NOT Boolean algebras
3. **Explicit construction**: $C(\mathbb{R}^n)$ (closed subspaces of n-dimensional real space) is both:
   - An orthomodular lattice
   - A commutative Girard quantale
   - Orthocomplement = linear negation

Inversions

Definition 1

An inversion on a poset $(P, \leq)$ is a map $(-)^{\circledperp} : P \to P$ such that:

- $x \leq y \Leftrightarrow x^{\circledperp} \geq y^{\circledperp}$ (order-reversing)
- $x^{\circledperp\circledperp} = x$ (involution)

Properties

An inversion is an involutive dual order automorphism (antitone involution).

Orthomodular Lattices

Definition 2

An orth

lattice $(X, \wedge, 1, {}^{\perp})$ is a meet semi-lattice with inversion ${}^{\perp}$ satisfying:

- $x^{\perp\perp} = x$ (involution)
- $x \leq y$ implies $y^{\perp} \leq x^{\perp}$ (order-reversing)
- $x \wedge x^{\perp} = 0$ (where $0 = 1^{\perp}$)

Definition 3

An ortholattice is orthomodular if:

$$x \leq y \text{ and } x^{\perp} \wedge y = 0 \implies x = y$$

Example 4

$C(H)$ = closed subspaces of Hilbert space H with:

- Meet = intersection ($\cap$)
- Orthocomplement = orthogonal complement

Residuated Posets and Quantales

Definition 5

A residuated poset $(P, \leq, \odot, \to, \leftarrow)$ satisfies:

$$x \odot y \leq z \quad \Leftrightarrow \quad x \leq y \to z \quad \Leftrightarrow \quad y \leq z \leftarrow x$$

- Integral: unit 1 is the greatest element.
- Unital: has unit e (not necessarily greatest element).

Definition 6

A quantale is a complete lattice Q with associative multiplication satisfying:

$$x \odot \bigvee_{i \in I} x_i = \bigvee_{i \in I} (x \odot x_i) \quad \text{and} \quad \left(\bigvee_{i \in I} x_i \right) \odot x = \bigvee_{i \in I} (x_i \odot x)$$

Girard Posets

Definition 7

An element d in a residuated poset is:

- Dualizing if: $d \leftarrow (x \rightarrow d) = x = (d \leftarrow x) \rightarrow d$.
- Cyclic if: $x \odot y \leq d \Leftrightarrow y \odot x \leq d$.

Definition 8

A Girard poset is a residuated poset with a cyclic dualizing element d.

Linear Negation

Define $x^{\oplus} = x \rightarrow d = d \leftarrow x$. Then P is unital with unit $e = d^{\oplus}$.

Example 9

Any complete Boolean algebra is a Girard quantale with $d = 0$.

Characterization Theorem

Theorem 10

Let $(P, \leq, \odot, \rightarrow, \leftarrow)$ be a unital residuated poset with unit e. The following are equivalent:

1. *P is a Girard poset*
2. *P has an inversion $^{\perp}$ with $x^{\perp} = x \rightarrow e^{\perp} = e^{\perp} \leftarrow x$*
3. *P has an inversion $^{\perp}$ with $t \odot x \leq y^{\perp} \Leftrightarrow y \odot t \leq x^{\perp}$*

Consequence

Cyclic dualizing element $d = e^{\perp}$ is uniquely determined by:

$$d = \bigvee_{x \in P} (x \odot x^{\perp})$$

Connection to Boolean Algebras

Proposition 11

Let $(P, \leq, \odot, \to, \leftarrow)$ be a Girard poset with inversion $^{\perp}$. Then P is a Boolean algebra if and only if:

- *P is an idempotent residuated lattice*
- *The cyclic dualizing element is $d = 0$*

Key Insight

In Boolean case: $x \odot x^{\perp} = x \wedge x^{\perp} = 0$ for all x

Complemented lattices

Definition 12

A bounded lattice $(X, \leq, \wedge, \vee, 0, 1)$ is *complemented* if, for every element $x \in P$, there exists an element $x' \in P$ (the complement of x) such that:

- $x \vee x' = 1$,
- $x \wedge x' = 0$.

Remark 13

- *A Boolean algebra is a complemented lattice that is also distributive.*
- *An ortholattice is a complemented lattice where the complementation is unique and satisfies additional properties like involution ($x'' = x$) and order-reversal ($x \leq y \Rightarrow y' \leq x'$).*

The Impossibility Result

Theorem 14

A complemented lattice admits an integral residuated structure if and only if it is a Boolean algebra

Proof Sketch

For integral residuated complemented lattice P:

1. $x \odot x' \leq x \wedge x' = 0$ (integrality)
2. $x = x \odot (x \vee x') = x \odot x$ (idempotency)
3. $x \odot y = x \wedge y$ (multiplication = meet)
4. P is distributive (multiplication distributes)
5. P is Boolean (complemented + distributive)

Corollary 15

A complemented lattice is an integral quantale iff it is a complete Boolean algebra

The Question

Theorem 14 tells us:

Integral case forces Boolean structure

Natural Question

Are there orthomodular lattices that are unital (but not integral) commutative quantales?

Answer

YES! We construct explicit examples using real coordinate spaces

The Construction: $C(\mathbb{R}^n)$

Theorem 16

For any $n \in \mathbb{N}$, the lattice $C(\mathbb{R}^n)$ of closed subspaces of $\mathbb{R}^n$ is:

1. *An orthomodular lattice with orthocomplement:*

$$U^{\perp} = \{(a_1, \ldots, a_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} a_i \cdot u_i = 0 \quad \text{for all } (u_1, \ldots, u_n) \in U\}$$

2. *A commutative Girard quantale with multiplication:*

$$S \odot T = \langle \{s \cdot t : s \in S, t \in T\} \rangle$$

(pointwise ring multiplication + linear span)

3. *The orthocomplement coincides with linear negation: $^{\perp} = ^{\odot}$*

Key Properties of $C(\mathbb{R}^n)$

Unital Structure

- Unit: $e = \langle\{(1, \dots, 1)\}\rangle$
- Dualizing element: $d = e^\perp = \{(a_1, \dots, a_n) \mid \sum a_i = 0\}$

Verification of Cyclicity

For subspaces $S, T, U \subseteq \mathbb{R}^n$:

$$S \odot T \subseteq U^\perp \Leftrightarrow \sum_{i=1}^{n} s_i \cdot t_i \cdot u_i = 0 \text{ for all } s \in S, t \in T, u \in U$$

$$\Leftrightarrow \sum_{i=1}^{n} u_i \cdot t_i \cdot s_i = 0 \text{ for all } s \in S, t \in T, u \in U$$

$$\Leftrightarrow U \odot T \subseteq S^\perp$$

Result

By Theorem 10, $C(\mathbb{R}^n)$ is a commutative Girard quantale!

Why This Example Matters

- **Non-Boolean**: For $n \geq 2$, $C(\mathbb{R}^n)$ is not distributive
- **Concrete**: Explicitly defined on familiar spaces
- **Unifying**: Bridges quantum logic and linear logic
  - Orthomodular structure (quantum)
  - Girard quantale structure (linear logic)
- **Natural**: Operations have geometric meaning
  - Meet = intersection
  - Orthocomplement = orthogonal complement
  - Multiplication = pointwise product span

Extension to Other Spaces

Complex Case

For matrix spaces $M_n(\mathbb{C})$, we have from Egger and Kruml [5]:

- $C(M_n(\mathbb{C}))$ is orthomodular
- $C(M_n(\mathbb{C}))$ is a non-commutative involutive Girard quantale
- For $n \geq 2$, this is also non-Boolean

Schatten Classes

For infinite-dimensional Hilbert spaces, similar constructions yield:

- Interval schemes of spectra organized like Łukasiewicz logic
- Orthomodular structure via Frobenius scalar product
- Non-commutative Girard quantales

Summary of Main Results

1. **Negative result**: Complemented + integral residuated ⇒ Boolean
2. **Characterization**: Three equivalent conditions for Girard posets with inversions
3. **Positive construction**: $C(\mathbb{R}^n)$ provides non-Boolean examples that are both:
    - Orthomodular lattices
    - Commutative Girard quantales
4. **Extensions**: Similar results for complex spaces and infinite-dimensional cases

Significance

Theoretical Impact

- Unifies quantum logic and linear logic frameworks
- Identifies precise boundary between Boolean and non-Boolean cases
- Provides concrete algebraic structures for hybrid systems

Open Questions

1. Does $C(\mathbb{C}^n)$ admit a compatible quantale structure?
2. What logic corresponds to orthomodular Girard quantales?
3. Can we develop sound and complete proof systems?
4. Are there natural examples beyond Hilbert space subspaces?

Key References I

[1] G. Birkhoff
Lattice Theory
Amer. Math. Soc., Providence, R.I., 1979

[2] G. Kalmbach
Orthomodular Lattices
Academic Press, London, 1983

[3] D. Kruml, J. Paseka
Algebraic and categorical aspects of quantales
Handbook of Algebra, vol. 5, Elsevier, 2008, pp. 323–362

[4] D. Kruml, J. Paseka
Nuclei and conuclei on Girard posets
Proc. EUSFLAT 2019, Atlantis Press, 2019, pp. 289–296

Key References II

[5] J.M. Egger, D. Kruml
Girard Couples of Quantales
Appl. Categor. Struct. 18, 123–133 (2010)

[6] K.J. Rosenthal
Quantales and Their Applications
Pitman Research Notes 234, Longman, 1990

Thank you for your attention!

Questions?

Contact: paseka@math.muni.cz