The Properties Defined by the Commutator in Algebras With Zero

Nebojša Mudrinski
joint work with Ivana Đurđev Brković

Department of Mathematics and Informatics, University of Novi Sad, Serbia
nmudrinski@dmi.uns.ac.rs
Mathematical Institute of the Serbian Academy of Sciences and Arts

AAA108, Vienna, Austria, February the 7^{th}, 2026

Commutators

Commmutators (Freese, McKenzie, Smith, Hagemann and Hermann)

Starting from groups generalized to algebras in congruence modular varieties

Higher commutators

In general, introduced by Bulatov in 2001 and further developed by Aichinger, Kearnes, Mayr, Moore, Moorhead, Opršal, Radović, Szendrei, Wires, $\sim$:

$$[\bullet, \ldots, \bullet]$$

as an n-ary operation, for each $n \geq 2$, on the lattice of congruences defined by the certain centralizing condition.

Definition

$[\alpha_1, \ldots, \alpha_n]$ is the smallest δ such that $C(\alpha_1, \ldots, \alpha_n; \delta)$.

Centralizing Relation

$C(\alpha_1, \ldots, \alpha_k, \beta; \delta)$, Bulatov 2001

Let $\mathbf{A}$ be an algebra. Let $k \geq 1$ and let $\alpha_1, \ldots, \alpha_k, \beta$ and δ be congruences in $\mathrm{Con}(\mathbf{A})$. Then we say that $\alpha_1, \ldots, \alpha_k$ *centralize* β *modulo* δ, and we write $C(\alpha_1, \ldots, \alpha_k, \beta; \delta)$, if for every polynomial $p \in \mathrm{Pol}(\mathbf{A})$, of arity $n_1 + \ldots + n_k + m$, where $n_1, \ldots, n_k, m \in \mathbb{N}$, and every $\mathbf{a}_1, \mathbf{b}_1 \in A^{n_1}$, ..., $\mathbf{a}_k, \mathbf{b}_k \in A^{n_k}$ and $\mathbf{c}, \mathbf{d} \in A^m$ such that

(1) $\mathbf{a}_j \ \alpha_j \ \mathbf{b}_j$, for $j = 1, \ldots, k$;

(2) $\mathbf{c} \ \beta \ \mathbf{d}$;

(3) $$\begin{gathered} p(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k, \mathbf{c}) \ \delta \ p(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k, \mathbf{d}) \\ p(\mathbf{a}_1, \mathbf{b}_2, \ldots, \mathbf{a}_k, \mathbf{c}) \ \delta \ p(\mathbf{a}_1, \mathbf{b}_2, \ldots, \mathbf{a}_k, \mathbf{d}) \\ \ldots \\ p(\mathbf{b}_1, \ldots, \mathbf{b}_{k-1}, \mathbf{a}_k, \mathbf{c}) \ \delta \ p(\mathbf{b}_1, \ldots, \mathbf{b}_{k-1}, \mathbf{a}_k, \mathbf{d}) \end{gathered}$$

we have $p(\mathbf{b}_1, \ldots, \mathbf{b}_k, \mathbf{c}) \ \delta \ p(\mathbf{b}_1, \ldots, \mathbf{b}_k, \mathbf{d})$.

The Properties Defined by Commutators

Definition

Let 1_A be the full and 0_A the equality relation of an algebra **A** and let $n \in \mathbb{N}$. Then **A** is

1. abelian if $[1_A, 1_A] = 0$;
2. n-nilpotent if $[1_A, 1_A]^{(n)} = 0$, where $[1_A, 1_A]^{(1)} = [1_A, 1_A]$ and $[1_A, 1_A]^{(k+1)} = [1_A, [1_A, 1_A]^{(k)}]$ for all $k \in \mathbb{N}$;
3. n-solvable if $[1_A]^{(n)} = 0_A$, where $[1_A, 1_A]^{(1)} = [1_A, 1_A]$ and $[1_A, 1_A]^{(k+1)} = [[1_A, 1_A]^{(k)}, [1_A, 1_A]^{(k)}]$ for all $k \in \mathbb{N}$;
4. n-supernilpotent if $[\underbrace{1_A, \dots, 1_A}_{n+1}] = 0_A$.

Remark

1-nilpotent = 1-solvable = 1-supernilpotent = abelian

General Problems

General questions

1. How are those properties connected?
2. Characterize those properties such that it would be easier to check them (in any sense)!

General Problems

General questions

1. How are those properties connected?
2. Characterize those properties such that it would be easier to check them (in any sense)!

From the fundamental properties of the commutators we know

1. abelian $\Rightarrow$ nilpotent $\Rightarrow$ solvable;
2. abelian $\Rightarrow$ supernilpotent.
3. In general, connection between nilpotent and supernilpotent property is not clear!

General Problems

General questions

1. How are those properties connected?
2. Characterize those properties such that it would be easier to check them (in any sense)!

From the fundamental properties of the commutators we know

1. abelian $\Rightarrow$ nilpotent $\Rightarrow$ solvable;
2. abelian $\Rightarrow$ supernilpotent.
3. In general, connection between nilpotent and supernilpotent property is not clear!

State of art

There has been a lot of contribution in Mal'cev algebras, in congruence modular varieties and in algebras with Taylor terms.

Semigroups and Semirings

Theorem (J. Radović and ∼, 2023)

Let $k \in \mathbb{N}$ and let S be a semigroup with zero o. TFAE:

(i) S is k-supernilpotent;

(ii) S is k-nilpotent;

(iii) $S^{k+1} = \{o\}$.

Definition

Here, $S^n := \{s_1 \cdot \ldots \cdot s_n \mid s_1, \ldots, s_n \in S\}$, for $n \in \mathbb{N}$.

Semigroups and Semirings

Theorem (J. Radović and $\sim$, 2023)

Let $k \in \mathbb{N}$ and let S be a semigroup with zero o. TFAE:

(i) S is k-supernilpotent;

(ii) S is k-nilpotent;

(iii) $S^{k+1} = \{o\}$.

Definition

Here, $S^n := \{s_1 \cdot \ldots \cdot s_n \mid s_1, \ldots, s_n \in S\}$, for $n \in \mathbb{N}$.

Theorem (M. Šobot and $\sim$, 2025)

Let $n \in \mathbb{N}$ and let $(S, +, \cdot, o)$ be a semiring. TFAE:

1. S is n-nilpotent;
2. S is additively cancellative and $S^{n+1} = \{o\}$;
3. S is n-supernilpotent.

The Main Goal

Our question

What are the properties of an algebra that force nilpotency and supernilpotency to be the same?

The Main Goal

Our question

What are the properties of an algebra that force nilpotency and supernilpotency to be the same?

Hypothesis

Is it having the zero element such a property?

On Algebras With Zero

Definition (Algebra with zero)

An algebra $\mathbf{A} = (A, F)$ is *algebra with zero* if there is a constant $0 \in F$ and for every operation $f \in F$ of arity $n \in \mathbb{N}$, we have $f(x_1, \ldots, x_n) = 0$ whenever there exist an $i \in \{1, \ldots, n\}$ such that $x_i = 0$.

On Algebras With Zero

Definition (Algebra with zero)

An algebra $\mathbf{A} = (A, F)$ is *algebra with zero* if there is a constant $0 \in F$ and for every operation $f \in F$ of arity $n \in \mathbb{N}$, we have $f(x_1, \ldots, x_n) = 0$ whenever there exist an $i \in \{1, \ldots, n\}$ such that $x_i = 0$.

Examples

semigroups with zero, groupoids with zero,...

On Algebras With Zero

Definition (Algebra with zero)

An algebra $\mathbf{A} = (A, F)$ is *algebra with zero* if there is a constant $0 \in F$ and for every operation $f \in F$ of arity $n \in \mathbb{N}$, we have $f(x_1, \ldots, x_n) = 0$ whenever there exist an $i \in \{1, \ldots, n\}$ such that $x_i = 0$.

Examples

semigroups with zero, groupoids with zero,...

Remark

Algebras with zero that have all fundamental operations of arity at most one are not interesting for the posed problems!

On Algebras With Zero

Definition (Algebra with zero)

An algebra $\mathbf{A} = (A, F)$ is *algebra with zero* if there is a constant $0 \in F$ and for every operation $f \in F$ of arity $n \in \mathbb{N}$, we have $f(x_1, \ldots, x_n) = 0$ whenever there exist an $i \in \{1, \ldots, n\}$ such that $x_i = 0$.

Examples

semigroups with zero, groupoids with zero,...

Remark

Algebras with zero that have all fundamental operations of arity at most one are not interesting for the posed problems!

Proposition

In algebras with zero that have all fundamental operations of arity at most one we have $[1_A, 1_A] = 0_A$.

Depending on the Argument

Definition

Let $\mathbf{A}$ be an arbitrary algebra. A polynomial $p : A^n \to A, n \in \mathbb{N}$

1. *syntactically depends on its i-th argument*, $i \in \{1, \ldots, n\}$ if x_i occurs in the corresponding polynomial term of p.
2. *semantically depends on its i-th argument*, $i \in \{1, \ldots, n\}$ if there exist $a, b \in A$ and $(x_1, \ldots, x_n) \in A^n$ such that

$$p(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n) \neq p(x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_n).$$

Remark

If a polynomial semantically depends on its argument then it syntactically depends on it. The opposite is not true in general! $\mathrm{Pol}_n^*(\mathbf{A})$ denotes the set of all n-ary polynomials over $\mathbf{A}$, that semantically depend on each of their arguments.

On polynomials in algebras with zero

Proposition

Let **A** be an algebra with zero 0. Then for each polynomial p of **A** of arity $n \in \mathbb{N}$ such that it depends syntactically on all arguments, we have $p(x_1, \ldots, x_n) = 0$ whenever there exist an $i \in \{1, \ldots, n\}$ such that $x_i = 0$.

On polynomials in algebras with zero

Proposition

Let $\mathbf{A}$ be an algebra with zero 0. Then for each polynomial p of $\mathbf{A}$ of arity $n \in \mathbb{N}$ such that it depends syntactically on all arguments, we have $p(x_1, \ldots, x_n) = 0$ whenever there exist an $i \in \{1, \ldots, n\}$ such that $x_i = 0$.

Proposition

In algebras with zero every nonzero polynomial semantically depends on its argument iff it syntactically depends on it.

On polynomials in algebras with zero

Proposition

Let **A** be an algebra with zero 0. Then for each polynomial p of **A** of arity $n \in \mathbb{N}$ such that it depends syntactically on all arguments, we have $p(x_1, \dots, x_n) = 0$ whenever there exist an $i \in \{1, \dots, n\}$ such that $x_i = 0$.

Proposition

In algebras with zero every nonzero polynomial semantically depends on its argument iff it syntactically depends on it.

Notation

For an algebra with zero **A** and $n \in \mathbb{N}$, $\mathrm{Pol}_n^*(\mathbf{A})$ denote the set of all n-ary polynomials over **A**, that depend on each of their arguments.

Ideals and Congruences

Definition

A nonempty set $I \subseteq A$ is an ideal of an arbitrary algebra $\mathbf{A} = (A, F)$ if for all $f \in F$ of arity $n \in \mathbb{N}$, we have $f(x_1, \ldots, x_n) \in I$, whenever there is an $i \in \{1, \ldots, n\}$ such that $x_i \in I$.

Ideals and Congruences

Definition

A nonempty set $I \subseteq A$ is an ideal of an arbitrary algebra $\mathbf{A} = (A, F)$ if for all $f \in F$ of arity $n \in \mathbb{N}$, we have $f(x_1, \ldots, x_n) \in I$, whenever there is an $i \in \{1, \ldots, n\}$ such that $x_i \in I$.

Remark

In an algebra with zero, the element zero belongs to every ideal!

Ideals and Congruences

Definition

A nonempty set $I \subseteq A$ is an ideal of an arbitrary algebra $\mathbf{A} = (A, F)$ if for all $f \in F$ of arity $n \in \mathbb{N}$, we have $f(x_1, \ldots, x_n) \in I$, whenever there is an $i \in \{1, \ldots, n\}$ such that $x_i \in I$.

Remark

In an algebra with zero, the element zero belongs to every ideal!

Rees congruence

$\rho_I = I \times I \cup \{(a, a) \mid a \in A \backslash I\}$ is a congruence of an arbitrary algebra $\mathbf{A}$ with ideal I.

Ideals and Congruences

Definition

A nonempty set $I \subseteq A$ is an ideal of an arbitrary algebra $\mathbf{A} = (A, F)$ if for all $f \in F$ of arity $n \in \mathbb{N}$, we have $f(x_1, \ldots, x_n) \in I$, whenever there is an $i \in \{1, \ldots, n\}$ such that $x_i \in I$.

Remark

In an algebra with zero, the element zero belongs to every ideal!

Rees congruence

$\rho_I = I \times I \cup \{(a, a) \mid a \in A \backslash I\}$ is a congruence of an arbitrary algebra $\mathbf{A}$ with ideal I.

Remark

In an algebra $\mathbf{A}$ with zero 0 we have: $\rho_{\{0\}} = 0_A$ and $\rho_A = 1_A$.

Commutators of Ideals

Definition

Let $n \geq 2$ and let $I_1, \ldots, I_n$ be ideals of an algebra $\mathbf{A}$. Then $[I_1, \ldots, I_n] := \{p(i_1, \ldots, i_n) \mid p \in \mathrm{Pol}_n^*(\mathbf{A}), i_1 \in I_1, \ldots, i_n \in I_n\}$.

Commutators of Ideals

Definition

Let $n \geq 2$ and let $I_1, \ldots, I_n$ be ideals of an algebra $\mathbf{A}$. Then $[I_1, \ldots, I_n] := \{p(i_1, \ldots, i_n) \mid p \in \mathrm{Pol}_n^*(\mathbf{A}), i_1 \in I_1, \ldots, i_n \in I_n\}$.

Proposition

Let $\mathbf{A}$ be an algebra and let $n \in \mathbb{N}$. Then $[I_1, \ldots, I_n]$ is an ideal of $\mathbf{A}$, for all ideals $I_1, \ldots, I_n$ of $\mathbf{A}$.

Commutators of Ideals

Definition

Let $n \geq 2$ and let $I_1, \ldots, I_n$ be ideals of an algebra $\mathbf{A}$. Then $[I_1, \ldots, I_n] := \{p(i_1, \ldots, i_n) \mid p \in \mathrm{Pol}_n^*(\mathbf{A}), i_1 \in I_1, \ldots, i_n \in I_n\}$.

Proposition

Let $\mathbf{A}$ be an algebra and let $n \in \mathbb{N}$. Then $[I_1, \ldots, I_n]$ is an ideal of $\mathbf{A}$, for all ideals $I_1, \ldots, I_n$ of $\mathbf{A}$.

Proposition

Let $k \geq 2$ and let $\rho_{I_1}, \ldots, \rho_{I_k}$ be Rees congruences on an arbitrary algebra $\mathbf{A} = (A, F)$. Then the condition $C(\rho_{I_1}, \ldots, \rho_{I_k}; \eta)$ is true for all congruences η of $\mathbf{A}$ such that $\rho_{[I_1, \ldots, I_k]} \leq \eta$.

Commutators in Algebras With Zero

Proposition

Let $k \geq 2$ and let $I_1, \ldots, I_k$ be ideals in an algebra **A** with zero. For the corresponding Rees congruences $\rho_{I_1}, \ldots \rho_{I_k}$ and $\theta \in \mathrm{Con}(\mathbf{A})$ such that $C(\rho_{I_1}, \ldots, \rho_{I_k}; \theta)$, we have $\rho_{[I_1, \ldots, I_k]} \subseteq \theta$.

Corollary

Let $n \geq 2$. For all ideals $I_1, \ldots, I_n$ of an algebra with zero **A** we have $[\rho_{I_1}, \ldots, \rho_{I_n}] = \rho_{[I_1, \ldots, I_n]}$.

Commutators in Algebras With Zero

Proposition

Let $k \geq 2$ and let $I_1, \ldots, I_k$ be ideals in an algebra $\mathbf{A}$ with zero. For the corresponding Rees congruences $\rho_{I_1}, \ldots \rho_{I_k}$ and $\theta \in \mathrm{Con}(\mathbf{A})$ such that $C(\rho_{I_1}, \ldots, \rho_{I_k}; \theta)$, we have $\rho_{[I_1, \ldots, I_k]} \subseteq \theta$.

Corollary

Let $n \geq 2$. For all ideals $I_1, \ldots, I_n$ of an algebra with zero $\mathbf{A}$ we have $[\rho_{I_1}, \ldots, \rho_{I_n}] = \rho_{[I_1, \ldots, I_n]}$.

Proposition

Let $\mathbf{A}$ be an algebra with zero. We denote by $\mathcal{I}(\mathbf{A})$ the algebra of all ideals on $\mathbf{A}$, with operations $\{\cap, \cup\} \cup \{[\cdot, \cdot], \ldots, [\cdot, \ldots, \cdot], \ldots\}$ and let $\mathbf{RCon}(\mathbf{A})$ denote the algebra of all Rees congruences corresponding to ideals in $\mathbf{A}$, with operations $\{\wedge, \vee\} \cup \{[\cdot, \cdot], \ldots, [\cdot, \ldots, \cdot], \ldots\}$. Then $\mathcal{I}(\mathbf{A}) \cong \mathbf{RCon}(\mathbf{A})$.

The Properties of the Commutator

Proposition

Let $\{\rho_{J_l} : l \in L\} \cup \{\rho_{I_0}, \rho_{I_1}, \ldots, \rho_{I_k}, \rho_J, \rho_{J_1}, \ldots, \rho_{J_k}\}$ be a family of Rees congruences and corresponding ideals in an algebra with zero **A**. Then, for $k, m \in \mathbb{N}$ we have

(HC4) $[\rho_{I_{\pi(1)}}, \ldots, \rho_{I_{\pi(k)}}] = [\rho_{I_1}, \ldots, \rho_{I_k}]$ for all $\pi \in S_k$;

(HC7) $\bigvee_{l \in L}[\rho_{I_1}, \ldots, \rho_{I_{j-1}}, \rho_{J_l}, \rho_{I_{j+1}}, \ldots, \rho_{I_k}] = [\rho_{I_1}, \ldots, \rho_{I_{j-1}}, \bigvee_{l \in L} \rho_{J_l}, \rho_{I_{j+1}}, \ldots, \rho_{I_k}]$;

(HC8) $[\rho_{I_1}, \ldots, \rho_{I_k}, [\rho_{I_{k+1}}, \ldots, \rho_{I_{k+m}}]] \leq [\rho_{I_1}, \ldots, \rho_{I_{k+m}}]$.

The Properties of the Commutator

Proposition

Let $\{\rho_{J_l} : l \in L\} \cup \{\rho_{I_0}, \rho_{I_1}, \ldots, \rho_{I_k}, \rho_J, \rho_{J_1}, \ldots, \rho_{J_k}\}$ be a family of Rees congruences and corresponding ideals in an algebra with zero **A**.Then, for $k, m \in \mathbb{N}$ we have

(HC4) $[\rho_{I_{\pi(1)}}, \ldots, \rho_{I_{\pi(k)}}] = [\rho_{I_1}, \ldots, \rho_{I_k}]$ for all $\pi \in S_k$;

(HC7) $\bigvee_{l\in L}[\rho_{I_1}, \ldots, \rho_{I_{j-1}}, \rho_{J_l}, \rho_{I_{j+1}}, \ldots, \rho_{I_k}] = [\rho_{I_1}, \ldots, \rho_{I_{j-1}}, \bigvee_{l\in L} \rho_{J_l}, \rho_{I_{j+1}}, \ldots, \rho_{I_k}]$;

(HC8) $[\rho_{I_1}, \ldots, \rho_{I_k}, [\rho_{I_{k+1}}, \ldots, \rho_{I_{k+m}}]] \leq [\rho_{I_1}, \ldots, \rho_{I_{k+m}}]$.

Proof

[(ACI4)] $[I_{\pi(1)}, \ldots, I_{\pi(k)}] = [I_1, \ldots, I_k]$ for all $\pi \in S_k$;
[(ACI7)] $\bigcup_{l\in L}[I_1, \ldots, I_{j-1}, J_l, I_{j+1}, \ldots, I_k] = [I_1, \ldots, I_{j-1}, \bigcup_{l\in L} J_l, I_{j+1}, \ldots, I_k]$;
[(ACI8)] $[I_1, \ldots, I_k, [I_{k+1}, \ldots, I_{k+m}]] \subseteq [I_1, \ldots, I_{k+m}]$.

The Properties

Proposition

Let **A** be an algebra with zero 0 and let $n \in \mathbb{N}$. Then **A** is

1. n-supernilpotent iff $[\underbrace{A, \ldots, A}_{n+1}] = \{0\}$;
2. n-nilpotent iff $[\underbrace{A, \ldots, [A}_{n}, A] \ldots] = \{0\}$

The Properties

Proposition

Let **A** be an algebra with zero 0 and let $n \in \mathbb{N}$. Then **A** is

1. n-supernilpotent iff $[\underbrace{A, \dots, A}_{n+1}] = \{0\}$;
2. n-nilpotent iff $[\underbrace{A, \dots, [A}_{n}, A] \dots] = \{0\}$

Proposition (I. Đ. Brković and $\sim$, 2026)

Let $n \in \mathbb{N}$. An algebra with zero **A** is n-supernilpotent iff all polynomials that depend on $n+1$ variables are zero polynomials.

The Properties

Proposition

Let **A** be an algebra with zero 0 and let $n \in \mathbb{N}$. Then **A** is

1. n-supernilpotent iff $[\underbrace{A, \ldots, A}_{n+1}] = \{0\}$;
2. n-nilpotent iff $[\underbrace{A, \ldots, [A}_{n}, A] \ldots] = \{0\}$

Proposition (I. Đ. Brković and ∼, 2026)

Let $n \in \mathbb{N}$. An algebra with zero **A** is n-supernilpotent iff all polynomials that depend on $n + 1$ variables are zero polynomials.

Proposition (I. Đ. Brković and ∼, 2026)

Let **A** be an algebra with zero and let $n \in \mathbb{N}$. If **A** is n-supernilpotent, then **A** is n-nilpotent.

Abelian Grupoids With Zero

Definition

An algebra $\mathbf{G} = (G, \cdot, 0)$ is *grupoid with zero* if $(G, \cdot)$ is a grupoid and $0 \in G$ such that $0 \cdot g = g \cdot 0 = 0$ for all $g \in G$.

Proposition

Let $\mathbf{G}$ be a groupoid with zero. Then, $\mathbf{G}$ is abelian if and only if it is a 0-groupoid.

Proof

abelian is 1-supernilpotence!

Nilpotent Grupoids With Zero

Definition

For all polynomial terms p, we define height, in abbreviation $ht(p)$ such that:

1. $ht(x) = ht(c) = 0$, if x is a variable or a constant c;
2. $ht(p_1 \cdot p_2) = max\{ht(p_1), ht(p_2)\} + 1$ for polynomial terms p_1 and p_2.

Nilpotent Grupoids With Zero

Definition

For all polynomial terms p, we define height, in abbreviation $ht(p)$ such that:

1. $ht(x) = ht(c) = 0$, if x is a variable or a constant c;
2. $ht(p_1 \cdot p_2) = max\{ht(p_1), ht(p_2)\} + 1$ for polynomial terms p_1 and p_2.

Notation

All polynomials with polynomial terms of height $n \in \mathbb{N}$ of an algebra $\mathbf{A}$ we denote by $\mathrm{Pol}^n(\mathbf{A})$.

Nilpotent Grupoids With Zero

Definition

For all polynomial terms p, we define height, in abbreviation $ht(p)$ such that:

1. $ht(x) = ht(c) = 0$, if x is a variable or a constant c;
2. $ht(p_1 \cdot p_2) = max\{ht(p_1), ht(p_2)\} + 1$ for polynomial terms p_1 and p_2.

Notation

All polynomials with polynomial terms of height $n \in \mathbb{N}$ of an algebra $\mathbf{A}$ we denote by $\mathrm{Pol}^n(\mathbf{A})$.

Theorem (I. Đ. Brković and $\sim$, 2026)

Let $n \in \mathbb{N}$. A grupoid $\mathbf{G}$ with zero 0 is n-nilpotent iff $\{p(G^m) \mid k, m \in \mathbb{N}, k \geq n, p \in \mathrm{Pol}_m^{k,*}(\mathbf{G})\} = \{0\}$.

Skinny Polynomials in Grupoids With Zero

Definition

The skiny polynomial of order 1 is (xy). A polynomial of a grupoid **G** is skinny of order $k \geq 2$ if its polynomial term has been obtained by finitely many applications of the following:

1. $((x_{\pi(1)}x_{\pi(2)})(x_{\pi(3)}x_{\pi(4)}))$ for all permutation $\pi \in S_4$ is skinny of order 2;
2. if p is skinny of order $k \geq 2$ then $(c \cdot p)$ and $(p \cdot c)$ are skinny of order k for all constants c;
3. if p and q are skinny of order $k - 1 \geq 1$ then $(p \cdot q)$ is skinny of order k.

Solvability in Groupoids With Zero

Theorem (I. Đ. Brković and $\sim$, 2026)

Let $\mathbf{G}$ be a groupoid with zero 0. Then, $\mathbf{G}$ is n-solvable, $n \in \mathbb{N}$ iff $\{p(G^{2^n}) \mid p \in \mathrm{Pol}^*_{2^n}(\mathbf{G}), p \text{ is skinny of order } n\} = \{0\}$.

Solvability in Groupoids With Zero

Theorem (I. Đ. Brković and ∼, 2026)

Let **G** be a groupoid with zero 0. Then, **G** is n-solvable, $n \in \mathbb{N}$ iff $\{p(G^{2^n}) \mid p \in \mathrm{Pol}^*_{2^n}(\mathbf{G}), p \text{ is skinny of order } n\} = \{0\}$.

Example of solvable but not nilpotent groupoid with zero:

$\cdot$	0	a	b	c	d	e
0	0	0	0	0	0	0
a	0	0	c	0	0	0
b	0	0	0	0	0	0
c	0	0	0	0	e	0
d	0	0	0	0	0	0
e	0	0	0	0	0	0

Connections of the Properties in Grupoids With Zero

Theorem (I. Đ. Brković and $\sim$, 2026)

A grupoid with zero is 2-nilpotent iff it is 2-supernilpotent.

Connections of the Properties in Grupoids With Zero

Theorem (I. Đ. Brković and ∼, 2026)

A grupoid with zero is 2-nilpotent iff it is 2-supernilpotent.

For $n \geq 3$

In groupoids with zero: n-nilpotent $\neq$ n-supernilpotent!

Connections of the Properties in Grupoids With Zero

Theorem (I. Đ. Brković and $\sim$, 2026)

A grupoid with zero is 2-nilpotent iff it is 2-supernilpotent.

For $n \geq 3$

In groupoids with zero: n-nilpotent $\neq$ n-supernilpotent!

Theorem (I. Đ. Brković and $\sim$, 2026)

A grupoid **G** with zero 0 is 3-supernilpotent iff it is 3-nilpotent and $(G^2)^2 = \{0\}$.

The Counterexample

Example

Let $n \geq 3$. Let $H_n = \{0, a_1, \ldots, a_{2^n-1}\}$ and $\mathbf{H}_n = (H_n, \cdot)$ where

$$x \cdot y = \begin{cases} a_i; & x = a_{2i}, y = a_{2i+1}, i \in \{1, \ldots, 2^{n-1} - 1\}; \\ 0; & \text{otherwise} \end{cases}$$

Proposition

$\mathbf{H}_n$ is n-nilpotent, but not n-supernilpotent!

Proof for $n = 4$

$((a_8 \cdot a_9) \cdot (a_{10} \cdot a_{11})) \cdot ((a_{12} \cdot a_{13}) \cdot (a_{14} \cdot a_{15})) = a_1 \neq 0$, but every polynomial of hieght 4 is zero!

Thank you for attention!