The class of congruence meet semidistributive varieties is not strong Maltsev

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Andrew Moorhead
Institute für Algbera
TU Dresden

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The interpretability lattice and Maltsev conditions

Definition

Let $\mathcal{V}_1 = \mathrm{Mod}(\Sigma_1)$ and $\mathcal{V}_2 = \mathrm{Mod}(\Sigma_2)$ be varieties over the signatures τ_1, τ_2, respectively. An *interpretation* of $\mathcal{V}_1$ in $\mathcal{V}_2$ is a mapping $I : \tau_1 \to \mathrm{Terms}(\tau_2)$ which preserves the satisfaction of Σ_1 identities, i.e.

$$f(x_1, \dots, x_n) \approx g(x_1, \dots, x_n) \in \Sigma_1 \implies \mathcal{V}_2 \models (If)(x_1, \dots, x_n) \approx (Ig)(x_1, \dots, x_n).$$

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This induces a preorder $\preceq$ on the class of varieties and we write $\mathcal{V}_1 \preceq \mathcal{V}_2$ to indicate that $\mathcal{V}_1$ interprets in $\mathcal{V}_2$. If both $\mathcal{V}_1 \preceq \mathcal{V}_2$ and $\mathcal{V}_2 \preceq \mathcal{V}_1$, we say $\mathcal{V}_1$ and $\mathcal{V}_2$ are *equi-interpretable*.

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- The variety $\mathcal{V}$ in the signature with a single ternary operation $m(xyz)$ axiomatized by the Maltsev identities $m(yxx) \approx m(xxy) \approx y$ interprets in the variety of groups, by mapping $m(xyz)$ to the term $xy^{-1}z$.

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- A class of varieties $\mathcal{C}$ is said to be characterized by a *strong Maltsev condition* if there exists a finitely presented (finite signature and finitely based) variety $\mathcal{V}$ such that $\mathcal{C} = \{\mathcal{W} : \mathcal{V} \preceq \mathcal{W}\}$.

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- A class of varieties $\mathcal{C}$ is said to be characterized by a *Maltsev condition* if there exists a countable sequence of finitely presented varieties $\mathcal{V}_1 \succeq \mathcal{V}_2 \succeq \cdots \succeq \mathcal{V}_i \succeq \dots$ such that $\mathcal{C} = \bigcup_{1 \leq i} \{\mathcal{W} : \mathcal{V}_i \preceq \mathcal{W}\}$.

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A variety $\mathcal{V}$ of algebras is *congruence meet semidistributive* if each congruence lattice of its members satisfies the implication

$$\gamma \wedge \alpha = \gamma \wedge \beta \implies \gamma \wedge (\alpha \vee \beta) = \gamma \wedge \alpha.$$

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TABLE 1. The six conditions

Type Omitting Class	Equivalent Property	Strong for l.f. varieties?	Strong in general?
$\mathcal{M}_{\{1\}}$	satisfies a nontrivial idempotent Maltsev condition	YES (Siggers)	YES (Olšák)
$\mathcal{M}_{\{1,5\}}$	satisfies a nontrivial congruence identity (see [17])	NO (KKVW)	NO
$\mathcal{M}_{\{1,4,5\}}$	congruence n-permutable, for some $n > 1$	NO (KKVW)	NO
$\mathcal{M}_{\{1,2\}}$	congruence meet semidistributive	YES (KKVW)	??
$\mathcal{M}_{\{1,2,5\}}$	congruence join semidistributive (see [17])	NO (KKVW)	NO
$\mathcal{M}_{\{1,2,4,5\}}$	congruence n-permutable for some n and congruence join semidistributive	NO (KKVW)	NO

Table taken from ‘Characterizations of several Maltsev conditions’ by Kozik, Krokhin, Valeriote, Willard.

The connection to the commutator and 2-congruences

Definition

Let $\mathbb{A}$ be an algebra and let θ_1, θ_2 be congruences of $\mathbb{A}$. We define the *algebra of (θ_1, θ_2)-matrices* as follows.

$$M(\theta_1, \theta_2) = \mathrm{Sg}_{A^{2^2}}\left(\left\{ \begin{array}{ccc} x & \text{---} & y \\ | & & | \\ x & \text{---} & y \end{array} : (x, y) \in \theta_1 \right\} \cup \left\{ \begin{array}{ccc} y & \text{---} & y \\ | & & | \\ x & \text{---} & x \end{array} : (x, y) \in \theta_2 \right\} \right).$$

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Informally, the above theorem is stating that the class of congruence meet semidistributive varieties is exactly the class of varieties which have no nontrivial abelian congruences, and the latter condition (2) holds for a variety $\mathcal{V}$ if and only if there are no abelian principle congruences, which is true if and only if the 'free' principle congruence for $\mathcal{V}$ is neutral.

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Let $S \leq A^{2^2}$. We define

- $H(S) := \left\{ \begin{array}{ccc} b & \text{---} & d \\ \vert & & \vert \\ a & \text{---} & c \end{array} : \exists e, f \left(\begin{array}{ccc} b & \text{---} & f \\ \vert & & \vert \\ a & \text{---} & e \end{array} , \begin{array}{ccc} f & \text{---} & d \\ \vert & & \vert \\ e & \text{---} & c \end{array} \in S \right) \right\}$ and
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In particular, $M(\theta, \theta) \subseteq \Delta(\theta, \theta)$, so it follows that $[\theta_1, \theta_2]_{TC} \leq [\theta_1, \theta_2]_H$.

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Theorem

Let $\mathbb{A}$ be an algebra and let θ be a congruence of $\mathbb{A}$. The following are equivalent.

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Theorem (Follows from a result of Kearnes and Szendrei)

Let $\mathbb{A}$ be a Taylor algebra and let α be a congruence of $\mathbb{A}$. Then

$$[\alpha, \alpha]_{TC} = [\alpha, \alpha]_H.$$

We define for a variety $\mathcal{V}$ and the (x, y)-*elementary matrices*:

$$E_{\mathcal{V}}(x,y) := \mathrm{Sg}_{\mathbb{F}_{\mathcal{V}}(x,y)^{2^2}}\left(\left\{ \begin{matrix} x & \text{—} & x \\ | & & | \\ x & \text{—} & x \end{matrix}, \begin{matrix} y & \text{—} & y \\ | & & | \\ y & \text{—} & y \end{matrix}, \begin{matrix} y & \text{—} & y \\ | & & | \\ x & \text{—} & x \end{matrix}, \begin{matrix} x & \text{—} & x \\ | & & | \\ y & \text{—} & y \end{matrix}, \begin{matrix} x & \text{—} & y \\ | & & | \\ x & \text{—} & y \end{matrix}, \begin{matrix} y & \text{—} & x \\ | & & | \\ y & \text{—} & x \end{matrix} \right\}\right).$$

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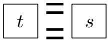
Theorem

Let $\mathcal{V}$ be a variety. The following are equivalent.

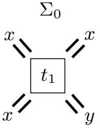
1. *$\mathcal{V}$ is congruence meet semidistributive,*
2. $\begin{matrix} x & \rule[0.5ex]{2em}{0.4pt} & x \\ \vert & & \vert \\ x & \rule[0.5ex]{2em}{0.4pt} & y \end{matrix} \in \Delta(\gamma,\gamma)$, *where γ is the congruence of the two generated free algebra $\mathbb{F}_{\mathcal{V}}(x,y)$ in $\mathcal{V}$ generated by the pair (x,y), and*
3. $\begin{matrix} x & \rule[0.5ex]{2em}{0.4pt} & x \\ \vert & & \vert \\ x & \rule[0.5ex]{2em}{0.4pt} & y \end{matrix} \in \mathrm{Cg}_2(E_{\mathcal{V}}(x,y)) = \bigcup_{n\geq 0}(V \circ H)^n(E_{\mathcal{V}}(x,y))$.

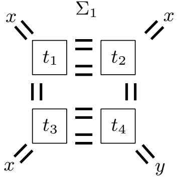


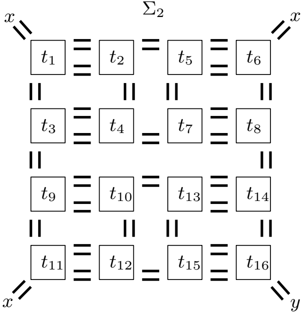
stands for

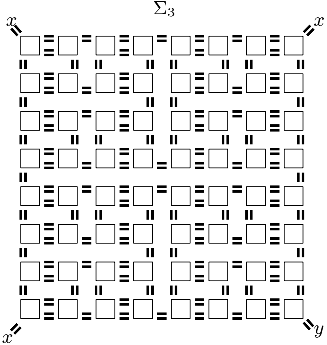
$$\begin{array}{ccccccc}
t(xxxyyy) & ___ & t(xxyyyx) & = & s(xxxyyy) & ___ & s(xxyyyx) \\
| & & | & & | & & | \\
t(xyxyxy) & ___ & t(xyyyxx) & = & s(xyxyxy) & ___ & s(xyyyxx)
\end{array}$$

The equational conditions $\Sigma_1, \Sigma_2, \ldots, \Sigma_i, \ldots$ which determine a Maltsev condition for congruence meet semidistributivity:









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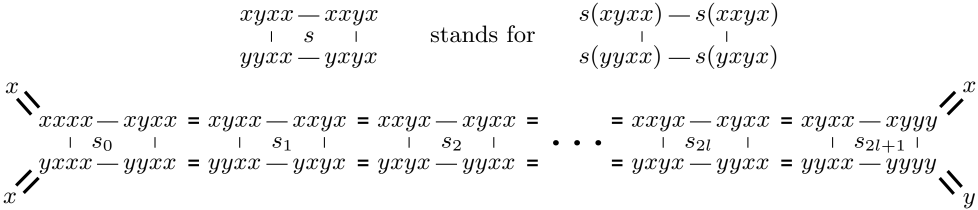

The condition Λ_l

Term analysis

Recall that

$$E_{\mathcal{W}_l}(x,y) := \mathrm{Sg}_{(\mathbb{F}_l)^{2^2}}\left(\left\{\begin{array}{ccc} x & \text{---} & x \\ | & & | \\ x & \text{---} & x \end{array}, \begin{array}{ccc} y & \text{---} & y \\ | & & | \\ y & \text{---} & y \end{array}, \begin{array}{ccc} y & \text{---} & y \\ | & & | \\ x & \text{---} & x \end{array}, \begin{array}{ccc} x & \text{---} & x \\ | & & | \\ y & \text{---} & y \end{array}, \begin{array}{ccc} x & \text{---} & y \\ | & & | \\ x & \text{---} & y \end{array}, \begin{array}{ccc} y & \text{---} & x \\ | & & | \\ y & \text{---} & x \end{array}\right\}\right).$$

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Our goal is to show that, for large enough l, there is no k where

$$\begin{array}{ccc} x & \text{---} & x \\ | & & | \\ x & \text{---} & y \end{array} \in (V \circ H)^n(E_k).$$

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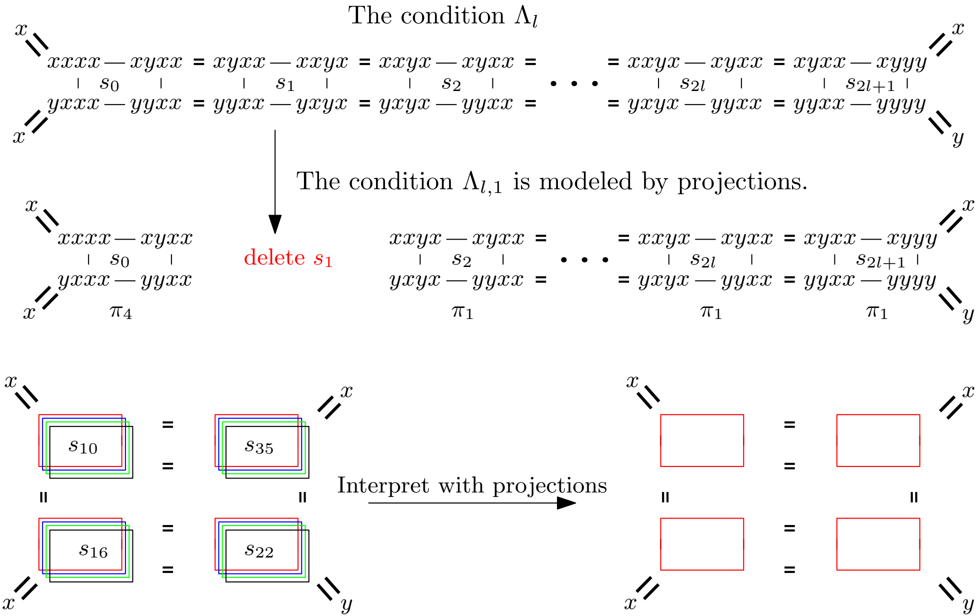
The condition Λ_l
$xxxx$ — $xyxx$ = $xyxx$ — $xxyx$ = $xxyx$ — $xyxx$ = … = $xxyx$ — $xyxx$ = $xyxx$ — $xyyy$
s_0 s_1 s_2 s_{2l} s_{2l+1}
$yxxx$ — $yyxx$ = $yyxx$ — $yxyx$ = $yxyx$ — $yyxx$ = … = $yxyx$ — $yyxx$ = $yyxx$ — $yyyy$
The condition $\Lambda_{l,1}$ is modeled by projections.
delete s_1
$xxxx$ — $xyxx$ s_0 $yxxx$ — $yyxx$ π_4
$xxyx$ — $xyxx$ = … = $xxyx$ — $xyxx$ = $xyxx$ — $xyyy$
s_2 s_{2l} s_{2l+1}
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π_1 π_1 π_1
s_{10} s_{35} s_{16} s_{22}
Interpret with projections
x y

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- We define a sequence of sets $\{x, y\} = F_l^0 \subseteq F_l^1 \subseteq \cdots \subseteq F_l^{k-1} \subseteq F_l^k \ldots$, where each F_l^k is the domain of a partial τ_l-algebra $\mathbb{F}_l^k$ with all τ_l operations defined on $(F_l^{k-1})^4$, for every $k \geq 1$.

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	$s_0^{\mathbb{F}_l^1}$	$s_1^{\mathbb{F}_l^1}$	$s_2^{\mathbb{F}_l^1}$	$\dots$	$s_{2l}^{\mathbb{F}_l^1}$	$s_{2l+1}^{\mathbb{F}_l^1}$
$xxxx$	x	x	x	$\dots$	x	x
$xxxy$	$s_0(xxxy)$	$s_1(xxxy)$	$s_2(xxxy)$	$\dots$	$s_{2l}(xxxy)$	$s_{2l+1}(xxxy)$
$xxyx$	$s_0(xxyx)$	$s_1(xxyx)$	$\leftarrow \mathbf{s_1(xxyx)}$	$\dots$	$s_{2l}(xxyx)$	$s_{2l+1}(xxyx)$
$xxyy$	$s_0(xxyy)$	$\leftarrow \mathbf{s_0(xxyy)}$	$s_2(xxyy)$	$\dots$	$s_{2l}(xxyy)$	$\leftarrow \mathbf{s_{2l}(xxyy)}$
$xyxx$	$s_0(xyxx)$	$\leftarrow \mathbf{s_0(xyxx)}$	$s_2(xyxx)$	$\dots$	$s_{2l}(xyxx)$	$\leftarrow \mathbf{s_{2l}(xyxx)}$
$xyxy$	$s_0(xyxy)$	$s_1(xyxy)$	$\leftarrow \mathbf{s_1(xyxy)}$	$\dots$	$s_{2l}(xyxy)$	$s_{2l+1}(xyxy)$
$xyyx$	$s_0(xyyx)$	$s_1(xyyx)$	$s_2(xyyx)$	$\dots$	$s_{2l}(xyyx)$	$s_{2l+1}(xyyx)$
$xyyy$	y	$s_1(xyyy)$	$s_2(xyyy)$	$\dots$	$s_{2l}(xyyy)$	x
$yxxx$	x	$s_1(yxxx)$	$s_2(yxxx)$	$\dots$	$s_{2l}(yxxx)$	y
$yxxy$	$s_0(yxxy)$	$s_1(yxxy)$	$s_2(yxxy)$	$\dots$	$s_{2l}(yxxy)$	$s_{2l+1}(yxxy)$
$yxyx$	$s_0(yxyx)$	$s_1(yxyx)$	$\leftarrow \mathbf{s_1(yxyx)}$	$\dots$	$s_{2l}(yxyx)$	$s_{2l+1}(yxyx)$
$yxyy$	$s_0(yxyy)$	$\leftarrow \mathbf{s_0(yxyy)}$	$s_2(yxyy)$	$\dots$	$s_{2l}(yxyy)$	$\leftarrow \mathbf{s_{2l}(yxyy)}$
$yyxx$	$s_0(yyxx)$	$\leftarrow \mathbf{s_0(yyxx)}$	$s_2(yyxx)$	$\dots$	$s_{2l}(yyxx)$	$\leftarrow \mathbf{s_{2l}(yyxx)}$
$yyxy$	$s_0(yyxy)$	$s_1(yyxy)$	$\leftarrow \mathbf{s_1(yyxy)}$	$\dots$	$s_{2l}(yyxy)$	$s_{2l+1}(yyxy)$
$yyyx$	$s_0(yyyx)$	$s_1(yyyx)$	$s_2(yyyx)$	$\dots$	$s_{2l}(yyyx)$	$s_{2l+1}(yyyx)$
$yyyy$	y	y	y	$\dots$	y	y

Input tuples $(a,b,c,d) \in F_l^k$ satisfy $\{a,b,c,d\} \cap (F_l^k \setminus F_l^{k-1}) \neq \emptyset$

	$s_0^{\mathbb{F}_l^{k+1}}$	$s_1^{\mathbb{F}_l^{k+1}}$	$s_2^{\mathbb{F}_l^{k+1}}$	$\dots$	$s_{2l}^{\mathbb{F}_l^{k+1}}$	$s_{2l+1}^{\mathbb{F}_l^{k+1}}$
$pppp$	p	p	p	$\dots$	p	p
$pppq$	$s_0(pppq)$	$s_1(pppq)$	$s_2(pppq)$	$\dots$	$s_{2l}(pppq)$	$s_{2l+1}(pppq)$
$ppqp$	$s_0(ppqp)$	$s_1(ppqp)$	$\leftarrow \mathbf{s_1(ppqp)}$	$\dots$	$s_{2l}(ppqp)$	$s_{2l+1}(ppqp)$
$ppqq$	$s_0(ppqq)$	$\leftarrow \mathbf{s_0(ppqq)}$	$s_2(ppqq)$	$\dots$	$s_{2l}(ppqq)$	$\leftarrow \mathbf{s_{2l}(ppqq)}$
$pqpp$	$s_0(pqpp)$	$\leftarrow \mathbf{s_0(pqpp)}$	$s_2(pqpp)$	$\dots$	$s_{2l}(pqpp)$	$\leftarrow \mathbf{s_{2l}(pqpp)}$
$pqpq$	$s_0(pqpq)$	$s_1(pqpq)$	$\leftarrow \mathbf{s_1(pqpq)}$	$\dots$	$s_{2l}(pqpq)$	$s_{2l+1}(pqpq)$
$pqqp$	$s_0(pqqp)$	$s_1(pqqp)$	$s_2(pqqp)$	$\dots$	$s_{2l}(pqqp)$	$s_{2l+1}(pqqp)$
$pqqq$	q	$s_1(pqqq)$	$s_2(pqqq)$	$\dots$	$s_{2l}(pqqq)$	p

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	$s_0^{\mathbb{F}_l^{k+1}}$	$s_1^{\mathbb{F}_l^{k+1}}$	$s_2^{\mathbb{F}_l^{k+1}}$	$\dots$	$s_{2l}^{\mathbb{F}_l^{k+1}}$	$s_{2l+1}^{\mathbb{F}_l^{k+1}}$
$abcd$	$s_0(abcd)$	$s_1(abcd)$	$s_2(abcd)$	$\dots$	$s_{2l}(abcd)$	$s_{2l+1}(abcd)$

Lemma

Let $l \geq 1$, and $m \geq 1$. Consider a set of τ_l-terms of the form

$$T = \{r_j(a_j, b_j, c_j, d_j) : r_j \in \tau_l \text{ and } a_j, b_j, c_j, d_j \in F_l, \text{ for } 1 \leq j \leq m\}$$

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and let

$$Z = \{z : \text{there exists } r_j(a_j, b_j, c_j, d_j) \in T \text{ with } r_j = s_z\}$$

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$$r_{j_1}^{\mathbb{F}_l}(a_{j_1}, b_{j_1}, c_{j_1}, d_{j_1}) = r_{j_2}^{\mathbb{F}_l}(a_{j_2}, b_{j_2}, c_{j_2}, d_{j_2}) \iff$$
$$r_{j_1}^{\mathbb{G}}(a_{j_1}, b_{j_1}, c_{j_1}, d_{j_1}) = r_{j_2}^{\mathbb{G}}(a_{j_2}, b_{j_2}, c_{j_2}, d_{j_2}),$$

where $\mathbb{G} = \mathbb{F}_{\mathcal{W}_{l,i}}(\bigcup_{1 \leq j \leq m}\{a_j, b_j, c_j, d_j\})$ is the algebra for $\mathcal{W}_{l,i}$ freely generated by

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Lemma

Let $l \geq 1$, *and* $m \geq 1$. *Consider a set of* τ_l*-terms of the form*

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$ppqp$	$s_0(ppqp)$	$s_1(ppqp)$	$\leftarrow \mathbf{s_1(ppqp)}$	$\dots$	$s_{2l}(ppqp)$	$s_{2l+1}(ppqp)$
$ppqq$	$s_0(ppqq)$	$\leftarrow \mathbf{s_0(ppqq)}$	$s_2(ppqq)$	$\dots$	$s_{2l}(ppqq)$	$\leftarrow \mathbf{s_{2l}(ppqq)}$
$pqpp$	$s_0(pqpp)$	$\leftarrow \mathbf{s_0(pqpp)}$	$s_2(pqpp)$	$\dots$	$s_{2l}(pqpp)$	$\leftarrow \mathbf{s_{2l}(pqpp)}$
$pqpq$	$s_0(pqpq)$	$s_1(pqpq)$	$\leftarrow \mathbf{s_1(pqpq)}$	$\dots$	$s_{2l}(pqpq)$	$s_{2l+1}(pqpq)$
$pqqp$	$s_0(pqqp)$	$s_1(pqqp)$	$s_2(pqqp)$	$\dots$	$s_{2l}(pqqp)$	$s_{2l+1}(pqqp)$
$pqqq$	q	$s_1(pqqq)$	$s_2(pqqq)$	$\dots$	$s_{2l}(pqqq)$	p

Intution: if $i \in Z$, where Z is the set of indices of basic τ_l-operation symbols being used, then we *at most* need to use identities involving s_{i-1}, s_i, and s_{i+1} to find the normal form for some $s_i(a, b, c, d)$.

Hence, for fixed n, there is obviously l large enough so that the lemma applies when $|Z| = 4^n$.

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$$\begin{array}{ccc} x & \text{---} & x \\ | & & | \\ x & \text{---} & y \end{array} \in \bigcup_{n \geq 0} (V \circ H)^n (E_{\mathcal{V}}(x, y)).$$

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We defined $E_0 = \left\{ \begin{array}{ccc} x & \text{---} & x \\ | & & | \\ x & \text{---} & x \end{array}, \begin{array}{ccc} y & \text{---} & y \\ | & & | \\ y & \text{---} & y \end{array}, \begin{array}{ccc} y & \text{---} & y \\ | & & | \\ x & \text{---} & x \end{array}, \begin{array}{ccc} x & \text{---} & x \\ | & & | \\ y & \text{---} & y \end{array}, \begin{array}{ccc} x & \text{---} & y \\ | & & | \\ x & \text{---} & y \end{array}, \begin{array}{ccc} y & \text{---} & x \\ | & & | \\ y & \text{---} & x \end{array} \right\}$ and

$$E_{k+1} = \{ r^{(\mathbb{F}_l)^{2^2}} (\alpha, \beta, \gamma, \delta) : r \in \tau_l \text{ and } \alpha, \beta, \delta, \gamma \in E_k \} \text{ for } k \geq 0.$$

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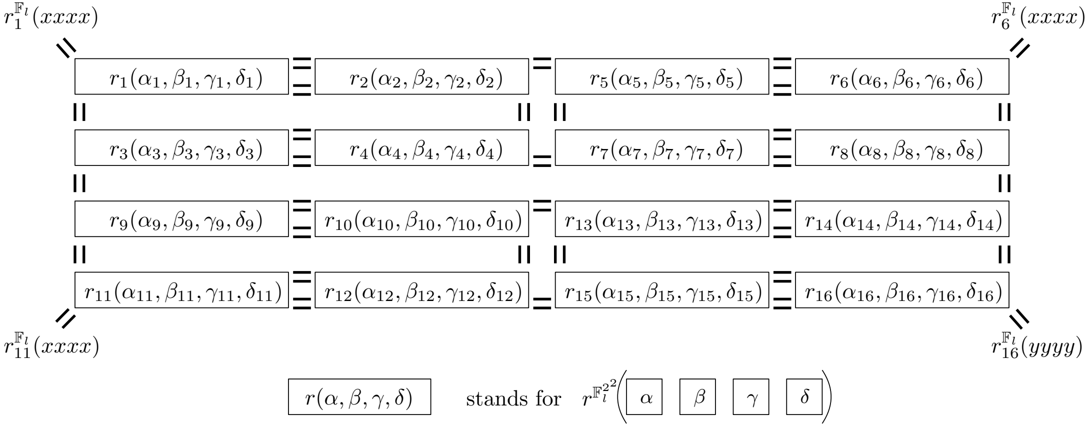
- Choose k minimal so that $\begin{array}{ccc} x & \text{---} & x \\ | & & | \\ x & \text{---} & y \end{array} \in \bigcup_{n \geq 0} (V \circ H)^n (E_k(x,y))$. Obviously, $k \neq 0$.

  We apply the lemma and find that $k-1$ works also, contradiction.


$r_1^{\mathbb{F}_l}(xxxx)$
$r_6^{\mathbb{F}_l}(xxxx)$
$r_1(\alpha_1, \beta_1, \gamma_1, \delta_1)$
$r_2(\alpha_2, \beta_2, \gamma_2, \delta_2)$
$r_5(\alpha_5, \beta_5, \gamma_5, \delta_5)$
$r_6(\alpha_6, \beta_6, \gamma_6, \delta_6)$
$r_3(\alpha_3, \beta_3, \gamma_3, \delta_3)$
$r_4(\alpha_4, \beta_4, \gamma_4, \delta_4)$
$r_7(\alpha_7, \beta_7, \gamma_7, \delta_7)$
$r_8(\alpha_8, \beta_8, \gamma_8, \delta_8)$
$r_9(\alpha_9, \beta_9, \gamma_9, \delta_9)$
$r_{10}(\alpha_{10}, \beta_{10}, \gamma_{10}, \delta_{10})$
$r_{13}(\alpha_{13}, \beta_{13}, \gamma_{13}, \delta_{13})$
$r_{14}(\alpha_{14}, \beta_{14}, \gamma_{14}, \delta_{14})$
$r_{11}(\alpha_{11}, \beta_{11}, \gamma_{11}, \delta_{11})$
$r_{12}(\alpha_{12}, \beta_{12}, \gamma_{12}, \delta_{12})$
$r_{15}(\alpha_{15}, \beta_{15}, \gamma_{15}, \delta_{15})$
$r_{16}(\alpha_{16}, \beta_{16}, \gamma_{16}, \delta_{16})$
$r_{11}^{\mathbb{F}_l}(xxxx)$
$r_{16}^{\mathbb{F}_l}(yyyy)$
$r(\alpha, \beta, \gamma, \delta)$ stands for $r^{\mathbb{F}_l^{2^2}}\left(\alpha \; \beta \; \gamma \; \delta\right)$

Thank you for your attention!