A Pastime Study on Majority Functions

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Outline

- **Pastime** (*noun*) something that you enjoy doing when you are not working (SYN) **hobby**

  $\langle\langle$ Oxford Dictionary $\rangle\rangle$

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I Introduction

Notation

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$\mathcal{O}_k^{(n)}$: the set of n-variable functions from E_k^n into E_k

$$f : E_k \times \cdots \times E_k \longrightarrow E_k$$

$$\mathcal{O}_k = \bigcup_{n=1}^{\infty} \mathcal{O}_k^{(n)}$$

Majority Function

What is a majority function?

A function $f \in \mathcal{O}_k^{(3)}$ is a *majority function* if it satisfies

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A majority function $f \in \mathcal{O}_k^{(3)}$ is completely determined by the values of f on W_k.

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Theorem (I. G. Rosenberg, 1986)

Any minimal function on E_k is of one of the five types :

(1) unary function
(2) binary idempotent function
(3) ternary majority function
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Let $f \in O_k^{(3)}$ be a majority function. Then,

f generates a **minimal clone**

if and only if

$f \in \langle g \rangle$ for any **essentially 3-ary** $g \in \langle f \rangle$.

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The point is, we need to prove $f \in \langle g \rangle$ only for "**essentially 3-ary**" function g.

Hence, in search of **minimal** majority functions, we can stay only in the range of 3-ary functions.

II Main Part

Relations on Majority Functions

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(1) For $f, f' \in \mathbb{M}$, $f \to f'$ if $f' \in \langle f \rangle$.

(2) For $f, f' \in \mathbb{M}$, $f \leftrightarrow f'$ if $f' \in \langle f \rangle$ and $f \in \langle f' \rangle$.

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Fact

(1) For $f, f' \in \mathbb{M}$, $f \to f'$ iff $\langle f' \rangle \subseteq \langle f \rangle$.

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Clearly, $\leftrightarrow$ is an equivalence relation on $\mathbb{M}$.

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Definition

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Lemma

The relation $\Rightarrow$ is a partial order on $\mathbb{Q}$, that is, $(\mathbb{Q}, \Rightarrow)$ is a finite poset.

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Lemma

If S is a "minimal element" in $(\mathbb{Q}, \Rightarrow)$, then S consists of **minimal functions**, i.e., any f in S is a minimal function.

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Clearly, $f'(a, b, c) = \text{median}\{a, b, c\}$ and $f''(a, b, c) = \min\{a, b, c\}$ have the same property.

(2) Let $f \in \mathbb{M}$ be defined by

$$f(x, y, z) = \begin{cases} 0 & \text{if } 0 \in \{x, y, z\}, \\ 1 & \text{otherwise} \end{cases}$$

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This f easily admits a generalization as shown in the next slide.

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Let $f \in \mathbb{M}$ be defined by

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Then, as above, $[f]$ is a singleton and f is a minimal function.

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But, I can give a (fairly long) chain of classes in $\mathbb{Q}$.

The next lemma plays a key rôle in the rest of my talk.

Key Lemma

For $f, g \in \mathbb{M}$, **if** $g(W_k) \setminus f(W_k) \neq \emptyset$ **then** $f \not\rightarrow g$.

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Proof. Let $d \in g(W_k) \setminus f(W_k)$. There exists $(a, b, c) \in W_k$ such that $g(a, b, c) = d$.

For any $h \in \langle f \rangle$ we shall show $h \neq g$. Suppose h is expressed as

$$h(x, y, z) \approx f(T_1(x, y, z), T_2(x, y, z), T_3(x, y, z))$$

for $T_1, T_2, T_3 \in \langle f \rangle$. So, for $(a, b, c) \in W_k$

$$h(a, b, c) = f(T_1(a, b, c), T_2(a, b, c), T_3(a, b, c)).$$

(Cont.)

For $(a, b, c) \in W_k$, we have $g(a, b, c) = d$ and

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Case 1. The values of $T_1(a, b, c)$, $T_2(a, b, c)$, $T_3(a, b, c)$ are pairwise distinct:

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Case 4. $T_i(a,b,c) = T_j(a,b,c)$ and T_i and T_j are variables:
Since $(a,b,c) \in W_k$, we must have the same variable (say, x) for T_i and T_j. Then h is a projection and so $h \neq g$. $\square$

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Later, we shall make use of the following easy corollaries.

Corollary 1

For $f, g \in \mathbb{M}$, **if** $f(W_k) \subset g(W_k)$ **then** $f \not\to g$.

Corollary 2

For $f, g \in \mathbb{M}$,

if $g(W_k) \setminus f(W_k) \neq \emptyset$ and $f(W_k) \setminus g(W_k) \neq \emptyset$

then $f \not\to g$ and $g \not\to f$.

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Definition

For $t \in E_k$ $(k > 2)$, let $f_t \in \mathbb{M}$ be the majority function which takes the following values on $(a, b, c) \in W_k$.

$$f_t(a,b,c) = \begin{cases} t & \text{if } \min\{a,b,c\} > t, \\ \min\{a,b,c\} - 1 & \text{if } 1 \leq \min\{a,b,c\} \leq t, \\ 0 & \text{if } \min\{a,b,c\} = 0 \end{cases}$$

[*Here, the natural order and the subtraction "$-$" are assumed on E_k.*]

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Note. $f_t(W_k) = \{u \in E_k \mid 0 \le u \le t\}$.

Example. Let $k > 5$. For $t = 5$, the definition of f_5 on W_k is

$$f_5(a,b,c) = \begin{cases} 5 & \text{if } \min\{a,b,c\} \geq 6, \\ \min\{a,b,c\} - 1 & \text{if } 1 \leq \min\{a,b,c\} \leq 5, \\ 0 & \text{if } \min\{a,b,c\} = 0 \end{cases}$$

and, therefore, the values of f_5 on W_k are as follows.

$$f_5(a,b,c) = \begin{cases} 5 & \text{if } \min\{a,b,c\} \geq 6, \\ 4 & \text{if } \min\{a,b,c\} = 5, \\ 3 & \text{if } \min\{a,b,c\} = 4, \\ 2 & \text{if } \min\{a,b,c\} = 3, \\ 1 & \text{if } \min\{a,b,c\} = 2 \\ 0 & \text{if } \min\{a,b,c\} = 0, 1 \end{cases}$$

Lemma

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Proof. Easy. If you watch the following for **2 minutes**, you would agree.

$$f_t\,(a, b, c) = \begin{cases} t & \text{if } \min\{a, b, c\} > t, \\ \min\{a, b, c\} - 1 & \text{if } 1 \leq \min\{a, b, c\} \leq t, \\ 0 & \text{if } \min\{a, b, c\} = 0, \end{cases}$$

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Notation

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Proof.

- By the above lemma,

$$f_{k-1} \to f_{k-2} \to \cdots \to f_2 \to f_1 \to f_0 .$$

- Since $f_{t-1}(W_k) \subset f_t(W_k)$, Corollary 1 of Key Lemma yields

$$f_0 \not\to f_1 \not\to f_2 \not\to \cdots \not\to f_{k-2} \not\to f_{k-1} .$$

□

Shape of $\mathbb{Q}$ (2)

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We shall show the existence of a set of **mutually incomparable classes** in $\mathbb{Q}$, whose size is **exponential** of k.

Recall:

Corollary 2

For $f, g \in \mathbb{M}$,
if $g(W_k) \setminus f(W_k) \neq \emptyset$ and $f(W_k) \setminus g(W_k) \neq \emptyset$
then $f \not\to g$ and $g \not\to f$.

For convenience, suppose k is even.

Let $\mathcal{H} = \{U \in \mathcal{P}(E_k) \mid |U| = k/2\}$.
For each $U \in \mathcal{H}$, take $f_U \in \mathbb{M}$ which satisfies $f_U(W_k) = U$.

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Lemma

For any $U, U' \in \mathcal{H}$,
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Proof. Since $|U| = |U'|$ $(= k/2)$, there is no inclusion relation between U and U' unless $U = U'$.
Therefore, by Corollary 2, we have $f_U \not\to f_{U'}$ and $f_{U'} \not\to f_{U'}$,
which imply $[f_U] \not\Rightarrow [f_{U'}]$ and $[f_{U'}] \not\Rightarrow [f_U]$. □

Let $\mathcal{S} = \{\, [f_U] \in \mathbb{Q} \mid U \in \mathcal{H}\}$

Proposition

(1) *$\mathcal{S}$ is a set of mutually incomparable classes in $\mathbb{Q}$.*

(2) $|\mathcal{S}| \geq 2^{k/2}$.

Proof. (1) By the above lemma.

(2) Since $|\mathcal{S}| = |\mathcal{H}| = \binom{k}{k/2}$, the claim follows easily as

$$\binom{k}{k/2} = \frac{k!}{(k/2)!(k/2)!} = \frac{k(k-1)\cdots(k/2+1)}{(k/2)!}$$

$$= \prod_{r=0}^{k/2-!} \frac{k-r}{k/2-r} = 2^{k/2} \prod_{r=0}^{k/2-!} \frac{k-r}{k-2r} \geq 2^{k/2}$$

$\square$

Final remark

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Proof. Let R be any **linear order** on E_k. Then the **median function** m_R generates no other ternary function (except for projections), and, for different orders R and S, the functions m_R and m_S must be different - except when R is the inverse order of S. □

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Note. A trivial upper bound: $|\mathbb{M}| = k^{k(k-1)(k-2)}$.

Thank you

for

your attention !