A network construction of Polyadic VB-algebras

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Introduction

- Polyadic algebras was introduced by P.R.Halmos in [Hal54] as an algebraic model of $\mathcal{L}_{\omega\omega}$.
- Cylindric algebras was introduced by A.Tarski [Tar52] as a n-dimension relation algebra which also serve as an algebraic model of $\mathcal{L}^{=}_{nm}$.
- Not all polyadic algebras and cylindric algebras are representable [HLT85, Ság12] as set-algebra. The representable cylindric algebras are not finitely axiomatizable [Mon64].
- Polyadic VB-algebras is an algebraic model for variable-binding calculus [PS95].

- R. Maddux proposed a construction of 3-dimensional cylindric algebras from relation algebras via network [Mad78].
- I. Hodkinson gives a new construction of n-dimensional polyadic and cylindric algebra from atomic relation algebra in [Hod12].
- A network is a finite graph that “could be realized” inside some representation of polyadic algebras or cylindric algebras.

Question : How do we design a ”network” for polyadic VB-algebra ?

Setting

Give two sets I, J with $J \subseteq I$. We call a mapping $\sigma : I \to I$ a transformation of I.

Notation

- The identity transformation is denoted by ι;
- For $\sigma, \tau \in I^I$, $\sigma J \tau$ means that $\sigma(i) = \tau(i)$ for all $i \in J$. Also, we denote $\sigma(I \setminus J)\tau$ as $\sigma J_* \tau$;
- If $\sigma J_* \iota$, we say J *supports* σ.

Classical Polyadic Algebras – 1

Definition 1 (Halmos, 1956)

An (existential) quantifier on a Boolean algebra $\mathbf{A}$ is a mapping $\exists : A \to A$ such that

1. $\exists 0 = 0$;
2. $p \leq \exists p$;
3. $\exists(p \wedge \exists q) = \exists p \wedge \exists q$

Classical Polyadic Algebras – 2

Definition 2 ([Hal54])

A (quasi-)polyadic algebra is a quadruple $\langle A, I, S, \exists\rangle$ where **A** is a Boolean algebra, I is a set (for variables), $\exists : P(I) \to A^A$ is a mapping from subsets of I to quantifiers on **A**, and $S : I^I \to End(A)$ such that

- $\exists(\emptyset)p = p$ for all $p \in A$
- $\exists(J \cup K) = \exists(J) \circ \exists(K)$ for all subsets J, K of A
- $S(1_I) = 1_A$
- $S(\sigma)(S(\tau)) = S(\sigma\tau)$
- If $J \subset I$ and σ, τ are transformations on I such that $\sigma(I - J) = \tau(I - J)$ then $S(\sigma)\exists(J) = S(\tau)\exists(J)$
- If $J \subseteq I$ and τ is a transformation which is injective on $\tau^{-1}J$, then $\exists(J)S(\tau) = S(\tau)\exists(\tau^{-1}J)$

Classical Polyadic Algebras – 3

Some explanations :

- no substitutions of variables, no corresponding changes to the propositional functions.
- applying substitution $\sigma \circ \tau$ of variables in a propositional function should have the same effect as applying τ first and then applying σ.
- once a variable has been quantified, the replacement of that variable by another one has no further effect.
- once the variable has been replaced by another one, quantification on the replaced variable has no further effect.

Classical Polyadic Algebras – 4

How about equality ?

Definition ([Hal57])

$(A, I, S, \exists, e)$ is a polyadic algebra with equality (or, an e-algebra) whenever $(A, I, S, \exists)$ is a polyadic algebra and e is a binary predicate satisfying

- $e(i, i) = 1$ for all $i \in I$,
- $p \wedge e(i, j) \leq S(i/j)p$ for all $i, j \in I$ and $p \in A$.

The follow correspondence is from [Gal57] :

- Every polyadic algebra with equality is a cylindric algebra,
- In the presence of an infinite supply of variables and a local finiteness condition, cylindric algebras are polyadic algebras with equality,
- The correspondence is one-to-one.

Language

Let $\mathcal{L} = \langle \mathcal{O}, \Phi, \mathbf{P}, \mathit{Var}, \rho, \nu \rangle$ be a language where $\mathcal{O}$ is a set of non-binding (propositional) connectives, $\mathbf{P}$ is a set of relation symbols, Var is a set of variables, $\rho : \mathcal{O} \sqcup \Phi \sqcup \mathbf{P} \to \omega$ is an arity function, and $\nu : \Phi \to \omega + 1 \setminus \{0\}$ is the binding rank function.
For $\mathcal{L}_{\kappa\lambda}$, $\rho(\forall) =$, $\nu(\forall) = \lambda$

([PS95]).Let I be a nonempty set. A polyadic $\langle \mathcal{L}, I\rangle$- algebra $\mathbf{A}$ is of the form

$$\langle A, (\circ^{\mathbf{A}} : \circ \in \mathcal{O}), (Q^{\mathbf{A}} : Q \in \Phi), S^{\mathbf{A}}, P^{\mathbf{A}}\rangle$$

where $\circ^{\mathbf{A}} : A^n \to A$ if $\rho(\circ) = n$, $Q^{\mathbf{A}} : \mathcal{P}_\omega(I) \to A^A$, $S^{\mathbf{A}} : I^I \to End(\mathbf{A})$, and such that the following axioms are satisfied :

- $S^{\mathbf{A}}_{\iota} x = x$;
- $S^{\mathbf{A}}_{\sigma}(S^{\mathbf{A}}_{\tau} x) = S^{\mathbf{A}}_{\sigma\tau} x$, for all $\sigma, \tau \in I^I$;
- $S^{\mathbf{A}}_{\sigma}(\circ^{\mathbf{A}}(x_1, \ldots, x_{\rho(\circ)})) = \circ^{\mathbf{A}}(S^{\mathbf{A}}_{\sigma} x_1, \ldots, S^{\mathbf{A}}_{\sigma} x_n)$, for all $\circ \in \mathcal{O}$, $\sigma \in I^I$;
- $S^{\mathbf{A}}_{\sigma} Q^{\mathbf{A}}_{J} x = S^{\mathbf{A}}_{\tau} Q^{\mathbf{A}}_{J} x$ for all $q \in \Phi$, $J \subseteq_\omega I$, and $\sigma, \tau \in I^I$ such that $\sigma J_* \tau$;
- $Q^{\mathbf{A}}_{J} S^{\mathbf{A}}_{\sigma} x = S^{\mathbf{A}}_{\sigma} Q^{\mathbf{A}}_{\sigma^{-1}(J)} x$ for all $Q \in \Phi$, $J \subseteq_\omega I$, and $\sigma, \tau \in I^I$ such that σ is injective on $\sigma^{-1}(J)$.

Polyadic VB-Algebras – 1

Definition 3

A value $\mathcal{L}$-algebra $\mathbf{V}$ is of the form

$$\langle V, (\circ^{\mathbf{V}} : \circ \in \mathcal{O}), (Q^{\mathbf{V}} : Q \in \Phi) \rangle$$

where $\circ^{\mathbf{V}} : V^{\rho(\circ)} \to V$ is a $\rho(\circ)$-ary operation on V for each $\circ \in \mathcal{O}$, and $Q^{\mathbf{V}} : \mathcal{P}(V) \rightharpoonup V$ is a partial unary second-order operation on V for each $Q \in \Phi$.

Polyadic VB-Algebras – 2

Given a value $\mathcal{L}$-algebra $\mathbf{V}$, two sets X, I, and an assignment $P^{\mathbf{V}} : X^{\rho(P)} \to V$. A *partial functional polyadic* $\langle \mathcal{L}, I \rangle$*- algebra* over X, $\bar{\mathbf{V}}$, is a structure of the form

$$\langle V^{X^I}, (\circ^{\bar{\mathbf{V}}} : \circ \in \mathcal{O}), (Q^{\bar{\mathbf{V}}} : Q \in \Phi), S^{\bar{\mathbf{V}}}, P^{\bar{\mathbf{V}}} \rangle$$

where $\circ^{\bar{\mathbf{V}}} : (V^{X^I})^{\rho(\circ)} \to V^{X^I}$, $Q^{\bar{\mathbf{V}}} : \mathcal{P}_\omega(I) \to [V^{X^I}, V^{X^I}]$, and $S^{\bar{\mathbf{V}}} : I^I \to End(\mathbf{V})$ are defined as follows :

- $P^{\bar{\mathbf{V}}}(p) = P^{\mathbf{V}}(\bar{p})$ where $\bar{p} = p(\bar{k})_{\bar{k} \in I^{\rho(P)}}$
- $(\circ^{\bar{\mathbf{V}}}(p_1, \ldots, p_{\rho(\circ)}))(\vec{x}) = \circ^{\mathbf{V}}(p_1(\vec{x}), \ldots, p_{\rho(\circ)}(\vec{x}))$ for all $p_1, \ldots, p_{\rho(\circ)} \in V^{X^I}$ and $\vec{x} \in X^I$;
- $(Q_J^{\bar{\mathbf{V}}} p)(\vec{x}) = Q^{\mathbf{V}}(\{p(\vec{y}) : \vec{x} J_* \vec{y}\})$, for all $p \in V^{X^I}$, $J \subseteq_\omega I$, and $\vec{x}, \vec{y} \in X^I$;
- $(S_\sigma^{\bar{\mathbf{V}}} p)(\vec{x}) = p(\sigma_* x)$ where $(\sigma_* \vec{x})_i = (\vec{x})_{\sigma(i)}$ for all $\sigma \in I^{(I)}$ and $\vec{x} \in X^I$.

Polyadic VB-Algebras – 4

Definition 4

A subalgebra $\bar{\mathbf{U}}$ of $\bar{\mathbf{V}}$ such that $Q_J^{\bar{\mathbf{U}}} p$ is a total function from X^I to $\mathbf{V}$ is called a functional polyadic $\langle \mathcal{L}, I \rangle$- algebra.

In [PS95], Pigozzi and Salibra prove the following two theorems :

Theorem 5 (Pigozzi and Salibra, 1995)

Every functional polyadic $\langle \mathcal{L}, I \rangle$-algebra is a polyadic $\langle \mathcal{L}, I \rangle$- algebra.

Polyadic VB-Algebras – 5

For the converse, as the classical case in [Hal54], we need to make some restrictions. An element a of a polyadic $\langle \mathcal{L}, I \rangle$-algebra has a finite support $J \subseteq I$ if $S_\sigma a = S_\tau a$ for all $\sigma, \tau \in I^{(I)}$ such that $\sigma J \tau$. A polyadic $\langle \mathcal{L}, I \rangle$-algebra is locally finite if every element has a finite support.

Theorem 6 (Pigozzi and Salibra, 1995)

Every locally finite polyadic $\langle \mathcal{L}, I \rangle$-algebra $\mathbf{A}$ of infinite dimension is isomorphic to a functional polyadic $\langle \mathcal{L}, I \rangle$-algebra whose domain X is I and universe of value $\mathcal{L}$-algebra is A.

Network – 1

From [Hod12], we know that the polyadic VB-type structure can be defined as

$$(X^I, P^{\mathbf{V}}, J_*, \sigma_*)$$

To define a network, we need to assume that $\mathbf{V}$ is from an ideal determined variety. A $\mathbf{V}$-network over I is a map $\lambda_{X^I} : X^I \times X^I \to \mathrm{Con}(\mathbf{V})$ satisfying

- $\lambda_{X^I}(\bar{x}, \bar{x}) = 0$,
- $\lambda_{X^I}(\bar{x}, \bar{y}) = \lambda_{X^I}(\bar{y}, \bar{x})$,
- $\lambda_{X^I}(\bar{x}, \bar{z}) \leq \lambda_{X^I}(\bar{x}, \bar{y}) \vee \lambda_{X^I}(\bar{y}, \bar{z})$,
- $\lambda_{X^I}(\circ(\bar{x}_1, \ldots, \bar{x}_n), \circ(\bar{y}_1, \ldots, \bar{y}_n)) = \bigvee_{i \leq n} \lambda_{X^I}(\bar{x}_i, \bar{y}_i)$

Network – 2

Let $J \subseteq_\omega I$, $Eq(X^I)$ to be the set of equivalence relations on X^I, $\sigma_* : X^I \to X^I$ induced by $\sigma : I \to I$

- $\sim =_J \sim'$ is defined as $\bar{x} \sim \bar{y} \iff \bar{x} \sim' \bar{y}$ for all $\bar{x}, \bar{y} \in X^{I \setminus J}$,
- $H(\sim) = \{N \subseteq X^I : \bigcup(N/\sim) = N\}$
- $\lambda_N^\eta =_J \lambda_N^{\eta'}$ is defined as $\lambda_N^\eta = \lambda_N^{\eta'}$ on $X^{I \setminus J}$,
- $\bar{x} \sim_{\sigma_*} \bar{y}$ is defined via $\sigma_*(\bar{x}) \sim \sigma_*(\bar{y})$.

Network – 3

Consider the structure $\langle \{\eta = (\sim^\eta, \lambda^\eta_N : N \in H(\sim^\eta))\}, \equiv_J, \sigma_* \rangle$

- $\sim^\eta \in Eq(X^I)$
- $\eta \equiv_J \eta'$ iff $\sim^\eta =_J \sim^{\eta'}$ and $\lambda^\eta_N =_J \lambda^{\eta'}_N$
- $\sigma_*(\eta) = ((\sim^\eta)_{\sigma_*}, \lambda^{\sigma_*(\eta)}_N : N \in H(\sim^\eta_{\sigma_*}))$
  - $\lambda^{\sigma_*(\eta)}_N(i,j) = \lambda^\eta_{\sigma_{*,\sim^\eta(N)}}(\sigma_* i, \sigma_* j)$
  - $\sigma_{*,\sim^\eta(N)} = \{\bar{x} : \forall \bar{y}\ \bar{x} \sim^\eta \sigma_*(\bar{y}) \rightarrow \bar{y} \in N\}$

Network – 4

Theorem 7

The functional polyadic VB-algebra constructed from $\langle\{\eta = (\sim^\eta, \lambda^\eta_N : N \in H(\sim^\eta))\}, \equiv_J, \sigma_*\rangle$ *is a polyadic VB-algebra.*

Is this really a network construction in the sense of [Mad78] and [Hod12] ?

Network – 5

The construction can be modified to study MV-relation algebra [Pop05]. A MV-relation algebra $\mathbb{A}$ is a structure $\langle A, \oplus, \odot, ^{-}, 0, 1, ;, ', I\rangle$ such that

- $\langle A, \oplus, \odot, ^{-}, 0, 1\rangle$ is an MV-algebra
- $\langle A, ;, I\rangle$ is a monoid
- $(x; y) \odot z = 0$ iff $(x'; z) \odot y = 0$ iff $(z; y') \odot x = 0$
- $(x \oplus y)' = x' \oplus y'$

Network – 6

We assume that I to be finite. A MV-network over I is a map $\lambda_I : I \times I \to J_p(\mathbf{A})$ satisfying

- $\lambda_I(x, y) \leq 1$,
- $\lambda_I(x, y) = \lambda_I(y, x)'$,
- $\lambda_I(x, z) \leq \lambda_I(x, y); \lambda_I(y, z)$,

Adopt the construction above, we can construction a polyadic MV-algebra [Sch80].

More work

Generalize the network construction of polyadic algebras from

- Weakening relation algebras,[JŠ23, CR26]
- FL^2 algebras [GJ20].

$\surd$ Work in progress with Peter Jipsen.

For polyadic VB-algebras :

- Non-classical hyperdoctrine (work in progress based on [KP10b] and [Mar21]).
- Second-order equational logic [FH10], nominal universal algebra [KP10a].

Thank you !

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