Pseudo-Heyting Algebras

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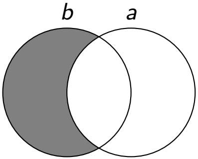
Difference and pseudo-Heyting implication



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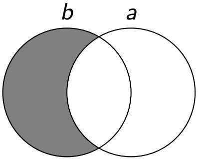

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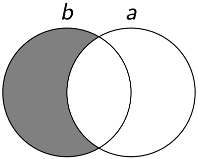

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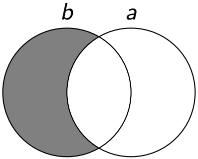

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An algebra $(L, \vee, \wedge, \rightsquigarrow, 0)$ is called A *pseudo-Heyting* if:

(a) $(L, \vee, \wedge, 0)$ is a lattice with zero,

(b) $a \wedge (a \rightsquigarrow b) = a \wedge b$,

(c) $a \wedge x = a \wedge b$ implies $x \leqslant a \rightsquigarrow b$.

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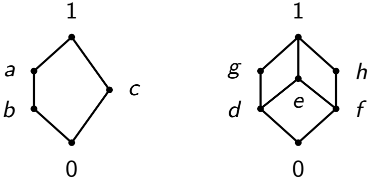

$$a \rightsquigarrow b = b, \quad c \rightsquigarrow 0 = a, \quad g \rightsquigarrow d = e, \quad g \rightsquigarrow 0 = h.$$

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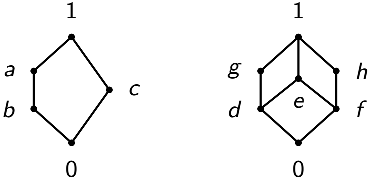

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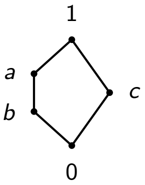

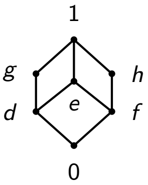

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Heyting implications $a \to b$ and $g \to d$ do not exist.

Fact.

(a) 1 always exists.

(b) If $a \to b$ exists, then $a \to b = a \rightsquigarrow b$.

(c) $a \rightsquigarrow 0$ is a pseudocomplement of a.

(d) Pseudo-Heyting algebras are $\wedge$-semidistributive.

(e) Complete pseudo-Heyting algebras are completely $\wedge$-semidistributive.

Sectionally pseudocomplemented lattices

A lattice $(L, \wedge, \vee, 1)$ is called *sectionally pseudocomplemented* [Chajda 2003] if for every $a, b \in L$ there exists:

$$a * b = \max\{x \in L : (a \vee b) \wedge x = b\}.$$

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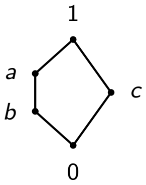

$$c * b = b, \quad c \rightsquigarrow b = a, \quad c \to b = a,$$

$$(x \vee y) \wedge (x * y) = y,$$

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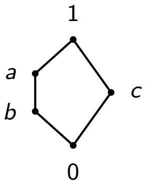

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Fact.

(a) If L is Heyting, then operations $\to$, $*$, $\rightsquigarrow$ coincide.

(b) The operations are mutually definable:

$$a * b = (a \vee b) \rightsquigarrow b, \quad \text{and} \quad a \rightsquigarrow b = a * (a \wedge b).$$

(c) Let L be a pseudo-Heyting algebra. Then L is sectionally pseudocomplemented. Moreover, L is Heyting iff $\rightsquigarrow$ and $*$ coincide.

Theorem [Chajda 2003]. A complete lattice L is pseudo-Heyting iff L is completely $\wedge$-semidistributive, i.e.,

$$(\forall b \in B)(a \wedge b = c) \;\Rightarrow\; a \wedge \bigvee B = c, \quad \text{for all } a, c \in L,\; B \subseteq L.$$

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Theorem [Chajda, Länger, Paseka 2019]. Sectionally pseudocomplemented lattices form a variety axiomatized by identities:

(a) lattice axioms,

(b) $z \vee y \leqslant x * ((x \vee y) \wedge (z \vee y))$,

(c) $(x \vee y) \wedge (x * y) = y$.

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Theorem. Pseudo-Heyting algebras form a variety axiomatized by identities:

(a) lattice axioms,

(b) $x \rightsquigarrow x = 1$,

(c) $x \wedge (x \rightsquigarrow y) = x \wedge y$,

(d) $x \rightsquigarrow (x \wedge y) = x \rightsquigarrow y$,

(e) $x \geqslant y \wedge z$ implies $x \wedge (y \rightsquigarrow z) = x \wedge ((x \wedge y) \rightsquigarrow (x \wedge z))$.

Theorem. Pseudo-Heyting algebra is Heyting iff one of the following holds:

(a) $z \leqslant x \rightsquigarrow y \;\Leftrightarrow\; x \wedge z \leqslant y$

(b) $x \leqslant y \;\Rightarrow\; z \rightsquigarrow x \leqslant z \rightsquigarrow y$,

(c) $(x \rightsquigarrow y) \wedge (y \rightsquigarrow z) \leqslant x \rightsquigarrow y$,

(d) $x \rightsquigarrow (y \rightsquigarrow z) = (x \wedge y) \rightsquigarrow z$,

(e) $x \rightsquigarrow (y \wedge z) = (x \rightsquigarrow y) \wedge (x \rightsquigarrow z)$,

(f) $(x \vee y) \wedge (x \rightsquigarrow y) = y$,

(g) $(x \vee y) \rightsquigarrow z = (x \rightsquigarrow z) \wedge (y \rightsquigarrow z)$,

(h) $(x \vee y) \rightsquigarrow y = x \rightsquigarrow y$,

(i) $x \rightsquigarrow (y \rightsquigarrow z) = (x \rightsquigarrow y) \rightsquigarrow (a \rightsquigarrow z)$,

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Theorem. Pseudo-Heyting algebra is complemented iff $x \vee (x \rightsquigarrow 0) = 1$ holds.

Theorem. Pseudo-Heyting algebra is Heyting iff one of the following holds:

(a) $z \leqslant x \rightsquigarrow y \iff x \wedge z \leqslant y$

(b) $x \leqslant y \Rightarrow z \rightsquigarrow x \leqslant z \rightsquigarrow y$,

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Theorem. Pseudo-Heyting algebra is complemented iff $x \vee (x \rightsquigarrow 0) = 1$ holds.

Theorem. Pseudo-Heyting algebra is Boolean iff one of the following holds:

(a) $(x \rightsquigarrow y) \rightsquigarrow y = x \vee y$,

(b) $x \rightsquigarrow y = y \iff x \vee y = 1$,

(c) $(x \rightsquigarrow 0) \rightsquigarrow 0 = x$.

Congruences

Fact. The variety of pseudo-Heyting algebras is arithmetical (congruence-distributive and congruence-permutable).

Proof follows from [Pixley 1963]: the term

$$m(x, y, z) = \big((x \vee y) \rightsquigarrow x\big) \wedge \big(x \vee z\big) \wedge \big((y \vee z) \rightsquigarrow z\big)$$

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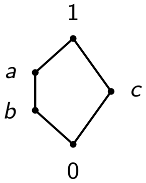

$(a,0) \in \Theta_c$: $\quad (a \rightsquigarrow 0) \wedge (0 \rightsquigarrow a) = c \wedge 1 = c,$

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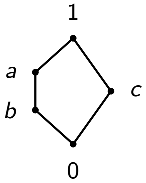

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Theorem. If c is standard and dually distributive, then Θ_c is a congruence.

Subdirectly irreducible algebras

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Fact. In pseudo-Heyting algebras

(a) ($\Leftarrow$) is true,

(b) ($\Rightarrow$) is not true: algebra $\mathfrak{N}_5$ (induced by lattice N_5) is subdirectly irreducible.
(More: every non-distributive cell is subdirectly irreducible.)

Subvarieties

Theorem. Let $\mathfrak{A}$ be pseudo-Heyting algebra. Then $\mathfrak{A}$ is Boolean if and only if it contains neither $\mathfrak{H}_3$ (the algebra induced by three element Heyting lattice H_3) nor $\mathfrak{N}_5$ as a subalgebra.

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Theorem. Let $\mathbb{V}$ be a subvariety of pseudo-Heyting algebras. Then:

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Theorem. There are continuum many subvarieties of pseudo-Heyting algebras that are not Heyting subvarieties.

Thank you for your attention!