Amalgamation property in quasivarieties of Sugihara algebras

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AAA, Vienna, 07.02.2026

This research was supported by the Czech Science Foundation project 25-18306M, INTERACT.

Amalgamation property for some varieties of algebras corresponding to non-classical logics

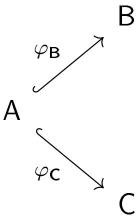
- Maksimova's 1977 characterization: Exactly eight subvarieties of Heyting algebras have the AP
- Gratzer Lakser 1971 exactly three subvarieties of pseudocomplemented distributive lattices have AP
- some classes of residuated lattices, particularly semilinear (Fussner Santschi, 2024, 2025)
- Sugihara monoids (Marchioni-Metcalfe 2012) exactly eight non-trivial varieties with AP.

Limitations and challenges

- known results restricted to varieties;
- no progress in terms of quasivarieties (quasivarieties correspond to consequence relations);
- Goal: get a deeper understanding of amalgamation-like properties in quasivarieites (interpolation-like properties for non-classical logic);
- Sugihara algebras as good case study (some nice properties, correspond historically well-known system of logic RM).

Amalgamation Property

class K has AP iff every span $\langle \varphi_B : A \hookrightarrow B, \varphi_C : A \hookrightarrow C\rangle$ in K can be completed in K (has an amalgam in K) i.e. there exists $D \in K$ and embeddings $\langle \psi_B : B \hookrightarrow D, \psi_C : C \hookrightarrow D\rangle$ such that $\psi_B\varphi_B = \psi_C\varphi_C$.



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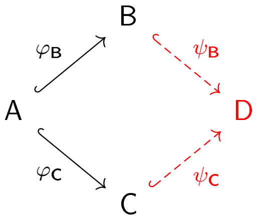
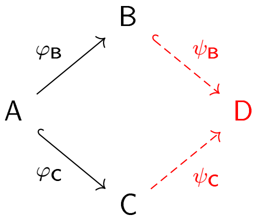


Transferable injections TIP – just diagramatically



A known algebraic fact:
TIP $\Leftrightarrow$ AP + RCEP

the quasivariety challenge

Results on amalgamation predominantly restricted to varieties.
Many known techniques are based on (R)CEP properties.
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Let Q *be any quasivariety with the* RCEP *such that* $\mathsf{Q}_{\mathrm{RFSI}}$ *is closed under subalgebras. Then* Q *has the* AP *if and only if every span of algebras in* $\mathsf{Q}_{\mathrm{FG}} \cap \mathsf{Q}_{\mathrm{RFSI}}$ *has an amalgam in* Q.

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(1) We usually have a bad handle on R(F)SI's in quasivarieties (lack of good characterizations)

(2) RCEP usually fails in (sub)quasivarieties even if the initial variety has it

Our result

Theorem

There are exactly five subquasivarieties (out of innfinitely continuum?) of Sugihara algebras with Amalgamation Property. Furthermore, AP$\Longrightarrow$RCEP for Sugihara quasivarieties.

Sugihara algebras

$\mathsf{Z} = \langle \mathsf{Z}, \wedge, \vee, \rightarrow, \neg \rangle$,

$$x \rightarrow y = \begin{cases} (-x) \vee y & x \leq y \\ (-x) \wedge y & x \nleq y. \end{cases}$$

Sugihara algebras are memebers of $\mathbb{V}(\mathsf{Z}) = HSP(\mathsf{Z})$.

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Sugihara algebras continued

It turns out that the lattice of subvarieties of SA forms a countable chain given by

$$\mathbb{V}(\mathsf{Z}_1) \subseteq \mathbb{V}(\mathsf{Z}_2) \subseteq \mathbb{V}(\mathsf{Z}_3) \subseteq \cdots \mathbb{V}(\mathsf{Z}) = \mathbb{V}(\mathsf{E}) = \mathsf{SA}.$$

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Further, $\mathrm{V}(\mathrm{Z}) = \mathbb{Q}(\mathrm{Z}) = \mathbb{Q}(\{\mathrm{Z}_n \mid n \geq 1\}) = \mathbb{Q}(\{\mathrm{Z}_{2n+1} \mid n \geq 0\})$, and also $\mathbb{Q}(\mathrm{E}) = \mathbb{Q}(\{\mathrm{Z}_{2n} \mid n \geq 1\})$ is a proper subquasivariety of SA.

Important results/useful lemmas

Lemma (Czelakowski-Dziobiak 1999)

All subquasiivarieties of Sugihara algebras with RCEP are the following:
$\mathbb{V}(\mathrm{Z}) \cup \{\mathbb{V}(\mathrm{Z}_n) : n \in \omega\} \cup \{\mathbb{Q}(\mathrm{E})\} \cup \{\mathbb{Q}(\mathrm{Z}_{2n}) : n \in \omega\}$.

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Lemma (K.K. 2022)

Let Q be a quasivariety such that $\mathbb{Q}(\mathrm{Z}_2) \subsetneq Q$. Then either $\mathrm{Z}_2 \times \mathrm{Z}_3 \in Q$ or $\mathrm{Z}_2 \times \mathrm{Z}_4 \in Q$.

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Lemma (Gil-Ferez et al. 2020)

The class of totally ordered odd Sugihara monoids has the amalgamation property.

Positive part

Lemma

Each of the quasivarieties $\mathbb{V}(\mathrm{Z}_2)$, $\mathbb{V}(\mathrm{Z}_3)$, $\mathbb{V}(\mathrm{Z})$, *and* $\mathbb{Q}(\mathrm{E})$ *has the amalgamation property.*

Easy part – all of them have RCEP, so we can apply the known techniques (transfer theorems). All (R)FSIs are chains – reducts of totally ordered Sugihara monoids.

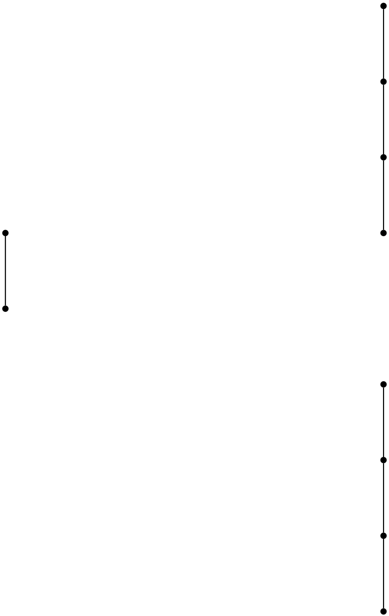
The challenging part– negative part

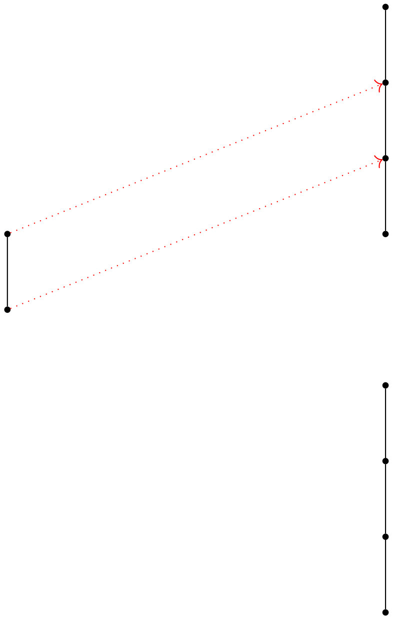
"Any other quasivariety does not have AP"
Our strategy is to use the so-called closure lemmas.

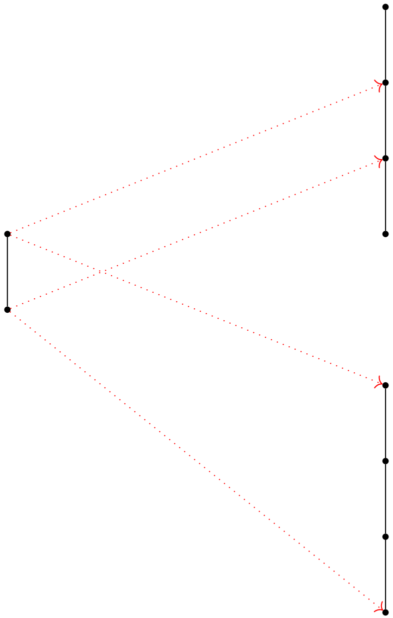
The two extending lemmas (even case)

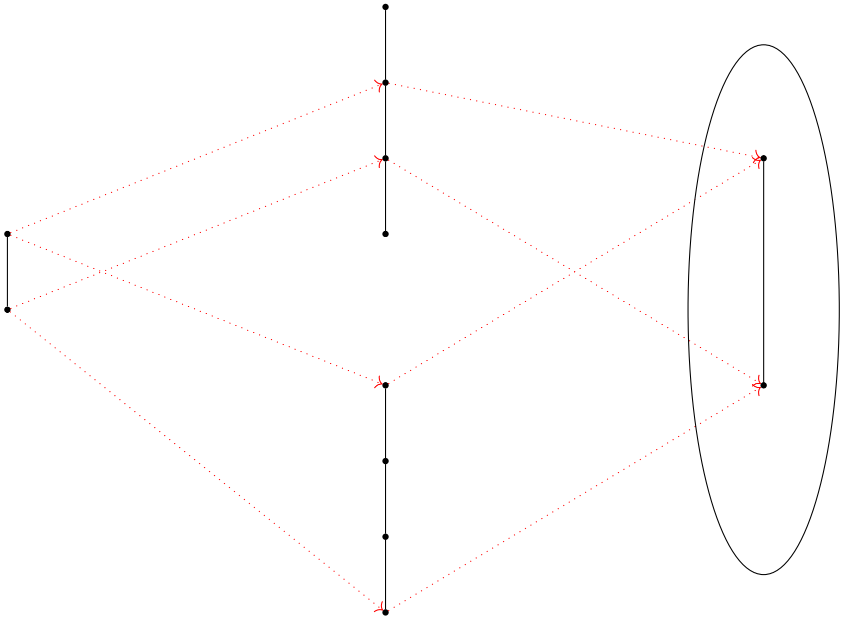
Lemma

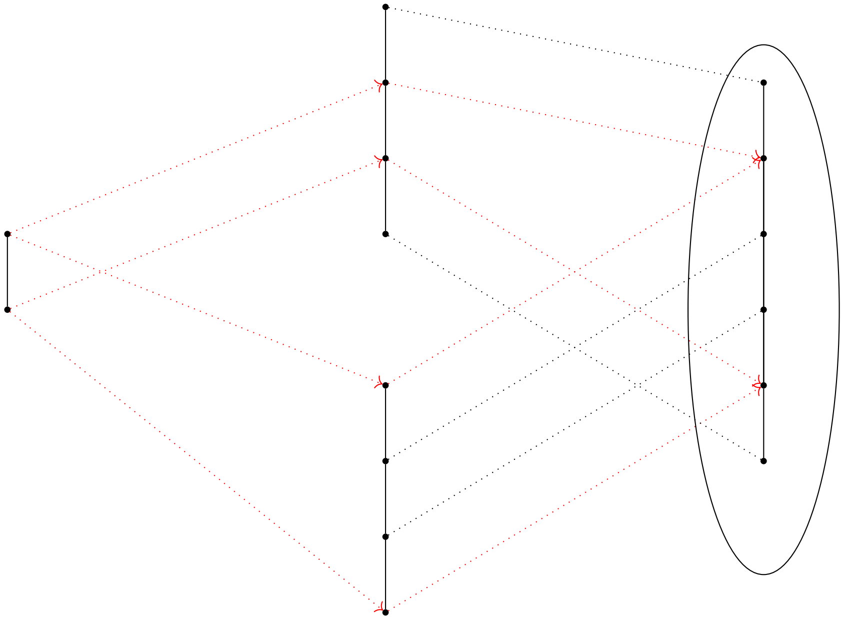
Assume Q *has AP. If* $Z_4 \in Q$, *then* $Z_{2n} \in Q$ *for every positive integer n. Consequently, if* $Z_4 \in Q$, *then* $E \in Q$.











The two extending lemmas (odd case)

Lemma

Assume Q *has AP. If* $\mathsf{Z}_3, \mathsf{Z}_4 \in \mathsf{Q}$, *then* $\mathsf{Q} = \mathsf{SA}$.

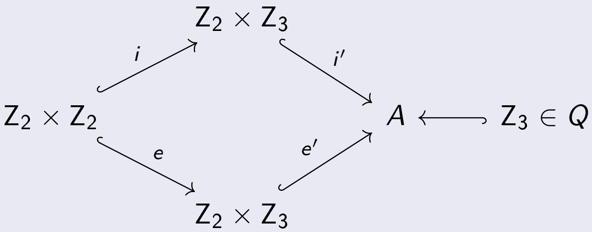
"The coordinate switch" embedding – odd and even cases

Lemma

Let Q *has AP. If* $Z_2 \times Z_3 \in Q$*, then* $Z_3 \in Q$.

Sketch of an argument.

Take a span $\langle i\colon Z_2 \times Z_2 \hookrightarrow Z_2 \times Z_3, e\colon Z_2 \times Z_2 \hookrightarrow Z_2 \times Z_3 \rangle$ where i is the identity embedding and e is the ‘coordinate-switch‘ embedding $(n, m) \mapsto (m, n)$.



Second 'switch embedding' closure lemma

Lemma

Let Q *has AP. If* $Z_2 \times Z_4 \in Q$*, then* $Z_4 \in Q$*.*

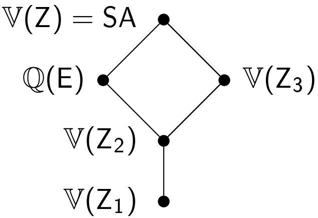
Theorem

Assume a nontrivial subquasivariety of SA, Q has AP. Then it is one of the four: $\mathbb{V}(Z_2)$, $\mathbb{V}(Z_3)$, $\mathbb{V}(Z)$, and $\mathbb{Q}(E)$.

proof idea

Since Q is nontrivial, $\mathbb{V}(Z_2) = \mathbb{Q}(Z_2) \subseteq Q$. Assume that this containment is proper. Then, by Lemma (K K 2022), either $Z_2 \times Z_3 \in Q$ or $Z_2 \times Z_4 \in Q$. We consider three mutually exclusive cases and end up in one of the three remaining quasivarieties: $\mathbb{V}(Z_3)$, $\mathbb{V}(Z)$, or $\mathbb{Q}(E)$

Poset of Qs with AP



comments/corollaries

1. In quasivarieties of Sugihara algebras $AP \Longrightarrow RCEP$. (Opposite direction is false) and additionally $AP \Leftrightarrow TIP$ (TIP always implies AP, but not the other way around)

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1. In quasivarieties of Sugihara algebras $AP \implies RCEP$. (Opposite direction is false) and additionally $AP \Leftrightarrow TIP$ (TIP always implies AP, but not the other way around)
2. There is only one proper subquasivariety of Sugiharas which has AP/TIP.

R – an axiomatic formulation

A1 $p \to p$

A2 $(p \to q) \to ((q \to r) \to (p \to r))$

A3 $p \to ((p \to q) \to q)$

A4 $(p \to (p \to q)) \to (p \to q)$

A5 $p \wedge q \to p$

A6 $p \wedge q \to q$

A7 $((p \to q) \wedge (p \to r)) \to (p \to q \wedge r)$

A8 $p \to p \vee q$

A9 $p \to q \vee p$

A10 $((q \to p) \wedge (r \to p)) \to (q \vee r \to p)$

A11 $p \wedge (q \vee r) \to (p \wedge q) \vee r$

A12 $(p \to \neg q) \to (q \to \neg p)$

A13 $\neg\neg p \to p$

The two rules of the system is modus ponens MP $\{\varphi, \varphi \to \psi\}/\psi$ and the adjunction rule AD $\{\varphi, \psi\}/\varphi \wedge \psi$.

R-mingle and Sugihara algebras

the logic R-mingle results from adding the 'mingle axiom' to basic system of relevance logic R
$p \to (p \to p)$.

Interpolation properties

If $\vdash \alpha \to \beta$, then there is a formula δ such that $\mathrm{var}(\delta) \subseteq \mathrm{var}(\alpha) \cap \mathrm{var}(\beta)$ and both $\vdash \alpha \to \delta$ and $\vdash \delta \to \beta$. (PCIP)

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RM fails the imperfect CIP

DIP, MIP and Robinson property

If $\Gamma \vdash \beta$ and $\mathrm{var}(\Gamma) \cap \mathrm{var}(\beta) \neq \emptyset$, then there is a set of formulas Δ such that $\mathrm{var}(\Delta) \subseteq \mathrm{var}(\Gamma) \cap \mathrm{var}(\beta)$ and both $\Gamma \vdash \Delta$ and $\Delta \vdash \beta$. (DIP)

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Whenever $X, Y \subseteq \mathrm{var}$ such that $X \cap Y \neq \emptyset$, T is a theory of L over X, and S is a theory of L over Y such that $T \cap \mathrm{Fm}_{\mathcal{L}}(X \cap Y) = S \cap \mathrm{Fm}_{\mathcal{L}}(X \cap Y)$, there exists a theory R of L over $X \cup Y$ such that $T = R \cap \mathrm{Fm}_{\mathcal{L}}(X)$ and $S = R \cap \mathrm{Fm}_{\mathcal{L}}(Y)$. (RP)

AAL characterization Czelakowski-Pigozzi 1999

AP $\Leftrightarrow$ RP
TIP $\Leftrightarrow$ MIP
Thus, as a corollary we have AP + RCEP $\Leftrightarrow$ MIP

logical corollaries

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2. In extensions of R–mingle $RP \Leftrightarrow MIP$
3. There is only one non-axiomatic extension of RM which has RP/MIP

Thank you!