Cacti Graphs and Majority Polymorphisms

AAA108

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Introduction

- We want to find out which CSPs actually have certain polymorphisms, to give a more concrete understanding of when these are tractable.
- In this talk, we present our work on the majority polymorphisms on cacti graphs.
- Based on ongoing work with Catarina Carvalho (University of Hertfordshire) and Barnaby Martin (Durham University).

Constraint Satisfaction Problem

The problem of asking whether a homomorphism between two general relational structures (say G and H) exists is known as a **constraint satisfaction problem** (CSP). For a fixed template H, we call this problem **CSP(H)**.

From here on out, instead of talking of relational structures, we will discuss CSPs on graphs.

Polymorphism

To categorise which CSPs are tractable, we need to look at higher symmetries of structures- polymorphisms.
Let $H^k = H \times H \times ... \times H$, be the categorical product of k copies of $H = (V(H), E(H))$.

Definition

A **polymorphism of** H **of arity** k is a homomorphism from H^k to H for $k \geq 2$. Namely, a mapping $f : V(H)^k \to V(H)$ is a polymorphism if and only if $f(u_1, ..., u_k)f(v_1, ..., v_k) \in E(H)$, for all $u_1v_1, u_2v_2, ..., u_kv_k \in E(H)$.

The Majority Polymorphism

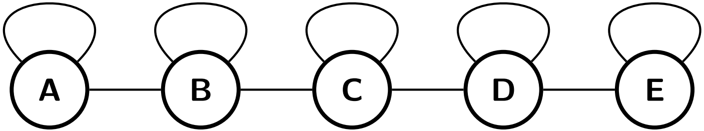
The Majority

Let m be a ternary polymorphism with the following condition:
$m : V(H^3) \to V(H)$
For all $x, y \in V(H)$, $m(x,x,y) = m(x,y,x) = m(y,x,x) = m(x,x,x) = x$.

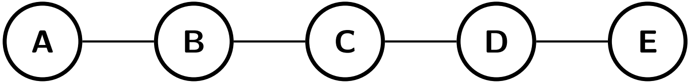
Theorem: Dalmau, Krokhin '08

If H admits a majority, then **CSP(H)** is in NL.

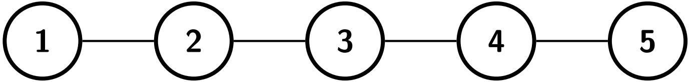
Examples



We can assign $m(x, y, z)$ to the vertex of x, y, z that is 'in the middle'.
$m(A, B, C) = B$
$m(B, C, D) = C$
etc.



How is this different? $m(A, B, C) \neq B$.



If the middle vertex has the same parity as at least 2 of the vertices we map this to the middle vertex.
If the middle vertex has the same parity of exactly one of the vertices we map this to the highest neighbour of the middle vertex.

Ref/Irref

Reflexive: Jawhari, Misane, Pouzet '86

A finite reflexive graph $H = (V(H), E(H))$ has an edge-preserving majority function if and only if H is a retract of a product of paths.

Irreflexive: Bandelt '92

A finite irreflexive graph $G = (V(G), E(G))$ is an absolute retract of bipartite graphs if and only if there exists an edge-preserving majority function on $V(G)$.

How can we extend these results to other graphs?

Partially Reflexive

Definition

A graph G is **loop-connected** if there exists a path of reflexive vertices between all pairs of reflexive vertices.

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We need to define the notion of 'in the middle'.

Definition

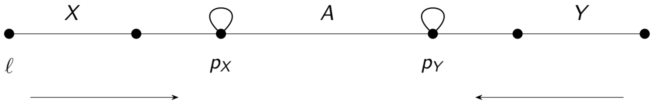
The **median**, $med\{u, v, w\} = x$, of three vertices u, v, w is a vertex x that lies on a shortest paths between each pair of u, v, w.

NB: This is not always defined and not always unique.

Definition

We define a **colour class** as a set of vertices that are independent and mapped to the same colour in a graph colouring. If $x, y \in V(G)$ and are in the same colour class then we write $[x] = [y]$.

Partially Reflexive Paths



We consider the following symmetric function, where $d(u,\ell) \leq d(v,\ell) \leq d(w,\ell)$.

$$m(u,v,w) = \begin{cases} v, & \text{if } (v \in A) \vee ([u]=[v] \wedge u,v \in X) \vee ([v]=[w] \wedge v,w \in Y); \\ w, & \text{if } ([u] \neq [v] \wedge w \in X); \\ p_X, & \text{if } ([u] \neq [v] \wedge v \in X \wedge w \notin X); \\ u, & \text{if } ([v] \neq [w] \wedge u \in Y); \\ p_Y, & \text{if } ([v] \neq [w] \wedge v \in Y \wedge u \notin Y). \end{cases}$$

We are essentially ‘filtering’ the values into the looped-component.

Extending

Trees

All loop-connected trees, T, have an edge preserving majority function $m : T^3 \to T$.

- This is done similarly to the path, with a descriptive majority function that is symmetric and non-conservative.
- The value of the majority is now not only based on the colour classes of the vertices, but also on the colour classes of the median of the three vertices.
- We define an order on the paths inside the trees that give a sensible definition of u, v, w.

Extending

Cycles

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Forbidden Cycles

- Any cycle of length 5 or higher.
- An irreflexive C_3.
- A partially reflexive C_4 with three looped vertices.
- A reflexive C_4.

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Pseudo-trees

We can then use the acceptable cycles and tree majority to craft an edge-preserving majority function for pseudo-trees (and pseudo-forests).

Finally...

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Theorem

A loop-connected cacti forest has a majority polymorphism if and only if it does not include one of the following as an induced subgraph:
i) Any cycle of length 5 or higher;
ii) An irreflexive C_3;
iii) A partially reflexive C_4 with three looped vertices;
iv) A reflexive C_4.

Thanks!