Quantum Polymorphisms of Relational Structures

Gideo Joubert

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Technische Universität Hambrug

Outline

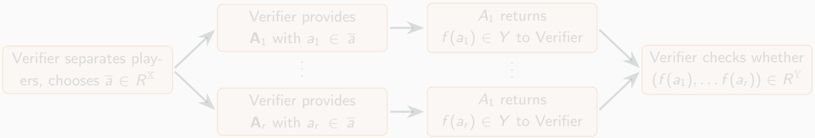
1. Quantum homomorphisms via games
2. Quantum polymorphisms
3. Boolean case: $\oplus$Pol vs. qPol

Based on *Quantum polymorphisms and commutativity gadgets*, joint work with Lorenzo Ciardo, Antoine Mottet, *arXiv:2511.23445* (preprint).

Homomorphism games

The homomorphism game

- Setup: Two known relational structures $\mathbb{X}$ and $\mathbb{Y}$ with a relation R of arity r. Cooperating players, $\mathbf{A}_1, \ldots, \mathbf{A}_r$; also a Verifier that runs the game.
- Players decide on a strategy f that takes domain elements from X and returns elements from Y; the game then proceeds as follows:



- Players win if $a_i = a_j \Rightarrow f(a_i) = f(a_j)$ and the check passes with a probability of 1.

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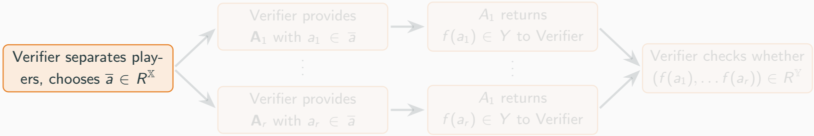

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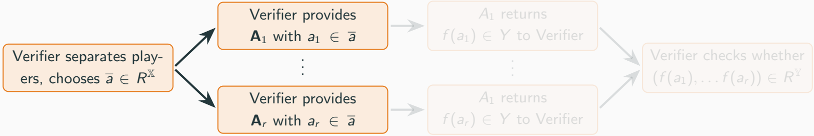

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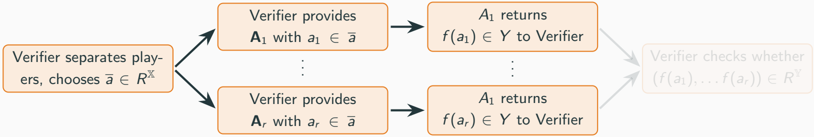

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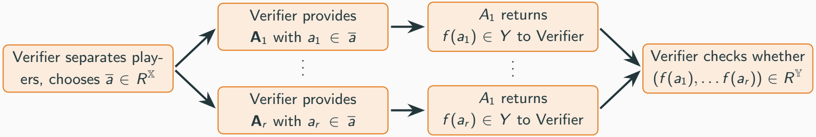

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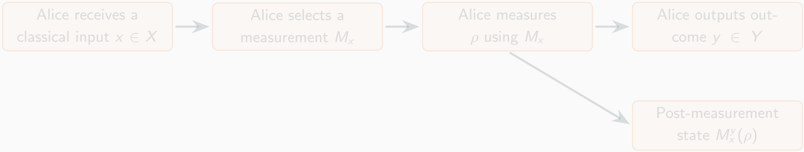
The quantum homomorphism game

A strategy Q is *quantum* if, during play; players may exploit some shared randomness through a shared quantum system $\rho \in H$, prepared by the players in the planning phase.

- A quantum strategy $Q : \mathbb{X} \to_q \mathbb{Y}$ consists of a set $\{M_x \mid x \in X\}$ of measurements on H that (WLOG) return elements from Y.



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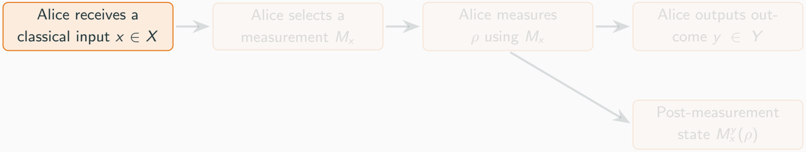

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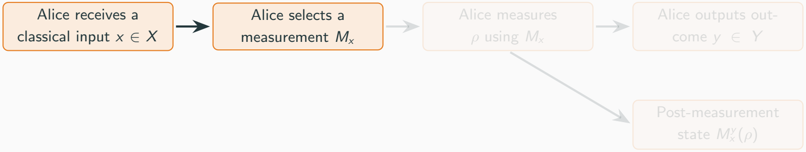

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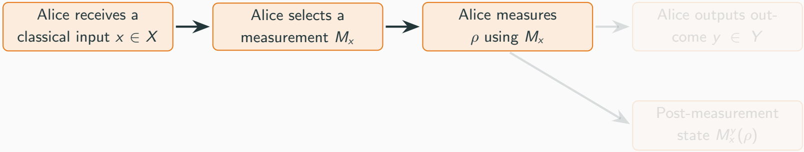

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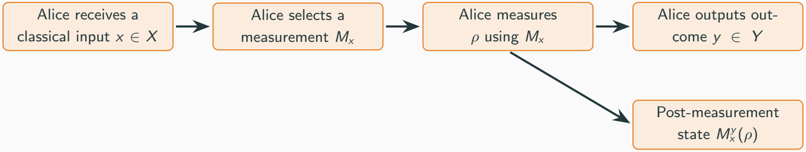

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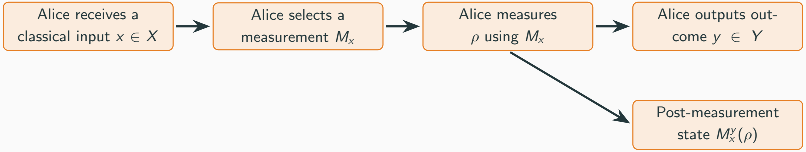

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A quantum function $Q\colon \mathbb{X} \to_q \mathbb{Y}$ over a Hilbert space H is a set $\{Q_{x,y} \mid x \in X, y \in Y\}$ of projection matrixes such that:

- $\sum_{y \in Y} Q_{x,y} = \mathrm{id}$.
- $Q_{x,y} Q_{x,y'} = 0$ for all $y \neq y'$ from Y.

Q is additionally a quantum homomorphism when:

(QH$_1$) $\prod_{i \in [r]} Q_{x_i,y_i} = 0$ if $\overline{x} = (x_1, \dots, x_n)$ is in $R^{\mathbb{X}}$ but, $\overline{y} \notin R^{\mathbb{Y}}$.

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Quantum polymorphisms

From homomorphisms to polymorphisms

Consider $Q\colon \mathbb{A} \to_H \mathbb{B}$ and $R\colon \mathbb{B} \to_{H'} \mathbb{C}$, we have canonical $R \bullet Q\colon \mathbb{A} \to_{H \otimes H'} \mathbb{C}$.

If we then take the quantum polymorphisms $Q\colon \mathbb{A}^n \to_q \mathbb{A}$ and consider $\mathrm{qPol}(\mathbb{A})$:

- $\mathrm{qPol}(\mathbb{A})$ is closed under composition.
- $\mathrm{qPol}(\mathbb{A})$ contains projections.

I.e. $\mathrm{qPol}(\mathbb{A})$ is a clone.

Lemma

Let $Q\colon \mathbb{X} \to_H \mathbb{A}$ and $Q'\colon \mathbb{X} \to_{H'} \mathbb{A}$ be quantum homomorphisms. Then $Q \oplus Q'$ is a quantum homomorphism $\mathbb{X} \to_{H \oplus H'} \mathbb{A}$.

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Definition

For any set $\mathcal{F}$ of operations, the *quantum closure* of $\mathcal{F}$ is the smallest quantum clone containing $\mathcal{F}$.

If $\mathcal{F}$ is classical, then quantum closure is just direct sums from $\langle \mathcal{F} \rangle$.

Defined by the property that any Q has $[Q_{x,y}, Q_{x',y'}] = 0$.

Quantum closing $\mathrm{Pol}(\mathbb{A})$, gives $\oplus\mathrm{Pol}(\mathbb{A})$, the (quantum) closure clone of $\mathbb{A}$.

Goal: In boolean setting, determine $\oplus\mathrm{Pol}(\mathbb{A})$ versus $\mathrm{qPol}(\mathbb{A})$.

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Quantum polymorphism example

Consider the boolean structure $\mathbb{B} := (\{0,1\}; S_{00}, S_{11}, S_{10})$

$\mathrm{Pol}(\mathbb{B})$ is the majority clone.

Claim: $\oplus\mathrm{Pol}(\mathbb{B}) \neq \mathrm{qPol}(\mathbb{B})$

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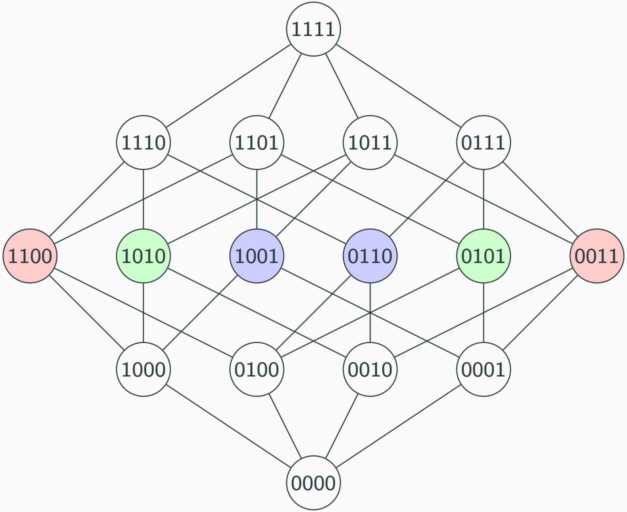

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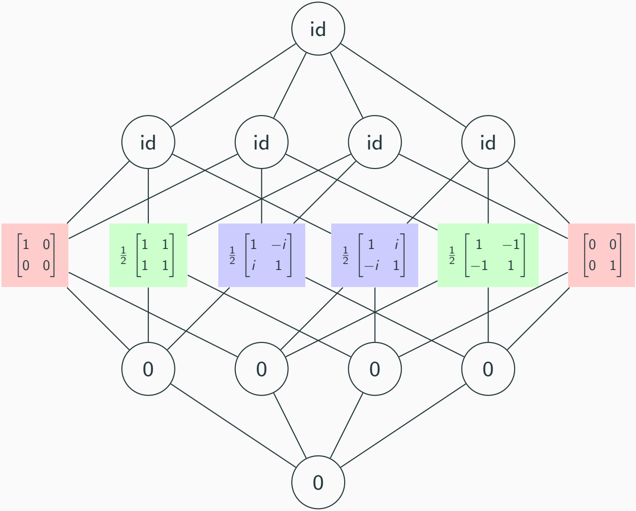

Boolean clones

Boolean relational structures

- Classical polymorphisms are understood.
- Domain $\{0,1\}$ so, $Q_{x,0} = \mathrm{id} - Q_{x,1}$ for any $Q\colon \mathbb{X} \to_q \mathbb{Y}$ with Y boolean. Moreover $\{0,1\}^n$ bijective to $\mathcal{P}[n]$, just map $S \subseteq [n]$ to $s \in \{0,1\}^n$ with $s_i = 1$ iff $i \in S$.
- Quantum polymorphism: $Q\colon \mathbb{X}^n \to_q \mathbb{X}$ given by $\{Q_S \mid S \in [n]\}$.

Polymorphism clone versus Closure clone

Theorem
If $\mathbb{A}$ is a Boolean relational structure such that $\mathrm{Pol}(\mathbb{A})$ *does not contain majority, then* $\mathrm{qPol}(\mathbb{A}) = \oplus\mathrm{Pol}(\mathbb{A})$.

We focus on the class $\mathbb{O}_t$ of boolean structures with relation given by $t \oplus r$ where $t \in \{0,1\}^k$ and $r \in R_{1/k}$ (the 1-in-k relation).

Example
If we take $k = 3$ and $t = (1,0,0)$ then

$$R_{1/3} = \{(1,0,0),(0,1,0),(0,0,1)\}$$

and so,

$$R_{(1,0,0)} = \{(0,0,0),(1,1,0),(1,0,1)\}.$$

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Proof sketch

Lemma

$\mathrm{qPol}(\mathbb{O}_{(1,0,0)}) = \oplus\mathrm{Pol}(\mathbb{O}_{(1,0,0)})$

Proof.

Consider $Q\colon \mathbb{O}^n_{(1,0,0)} \to_q \mathbb{O}_{(1,0,0)}$, $S, T \subseteq [n]$ with $S \cap T = \emptyset$ and let $R := S \cup T$. Then (r, s, t) is in $R_{(1,0,0)}$, which gives identities:

- $Q_{S\cup T} Q_S Q_T = 0$, since $(1,1,1)$ is not in $R_{(1,0,0)}$,
- $(\mathrm{id} - Q_{S\cup T}) Q_S Q_T = 0$, since $(0,1,1)$ is not in $R_{(1,0,0)}$,
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Then manipulate these identities to derive $Q_S = \sum_{i\in S} Q_{\{i\}}$ for any $S \subseteq [n]$. Further derive that $[Q_S, Q_T] = 0$, for every S, T and thus $\mathrm{qPol}(\mathbb{O}_{(1,0,0)}) = \oplus\mathrm{Pol}(\mathbb{O}_{(1,0,0)})$. □

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- $(\mathrm{id} - Q_{S\cup T}) Q_S Q_T = 0$, since $(0,1,1)$ is not in $R_{(1,0,0)}$,
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- $(\mathrm{id} - Q_{S\cup T})(\mathrm{id} - Q_S) Q_T = 0$, since $(0,0,1)$ is not in $R_{(1,0,0)}$.

Then manipulate these identities to derive $Q_S = \sum_{i\in S} Q_{\{i\}}$ for any $S \subseteq [n]$. Further derive that $[Q_S, Q_T] = 0$, for every S, T and thus $\mathrm{qPol}(\mathbb{O}_{(1,0,0)}) = \oplus\mathrm{Pol}(\mathbb{O}_{(1,0,0)})$. □

Proof sketch

Lemma

$\mathrm{qPol}(\mathbb{O}_{(1,0,0)}) = \oplus\mathrm{Pol}(\mathbb{O}_{(1,0,0)})$

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Proof sketch

Claim

$\mathrm{qPol}(\mathbb{O}_t) = \oplus\mathrm{Pol}(\mathbb{O}_t)$ *for any* $t \in 0, 1^k$.

Lemma

Every relational structure $\mathbb{A}$ *with a relation* R *that has any full binary projection has the property that* $\mathrm{qPol}(\mathbb{A}) = \oplus\mathrm{Pol}(\mathbb{A})$.

Theorem

For any relation $R \subseteq \{0, 1\}^k$ *the following statements are equivalent:*

1. R *is not invariant under majority, majority preserves every proper projection of* R*, and every binary projection of* R *is not full.*
2. $R = R_t$ *for some tuple* t.

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Thank you for listening!

Questions?