Congruence lattices of prime-cycled algebras

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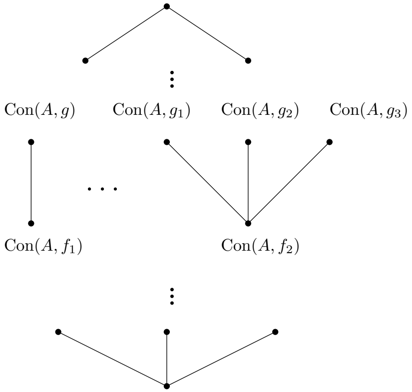
Introduction

- The set of all congruences on a algebra (A, F) (ordered by inclusion) forms a lattice, denoted by $\mathbf{Con(A, F)}$.
- The set of all congruence lattices of all algebras defined on a fixed base set A forms a lattice, denoted $\boldsymbol{\mathcal{E}_A}$. I.e. $\mathcal{E}_A = \{\mathrm{Con}(A, F) : F \subseteq A^A\}$
- Let L be a lattice. A nonunit element $a \in L$ is called **meet-irreducible** (shortly $\wedge$-irreducible) if $a = b_1 \wedge b_2$ implies $a \in \{b_1, b_2\}$.

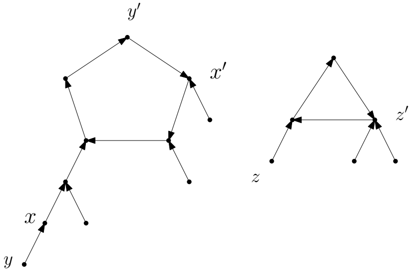
Introduction

We denote monounary algebra: (A, f)



Introduction

- Distance from cycle: $t_f(a)$
- For $x, y \in A$ let $\theta_f(x, y)$ denote the smallest congruence of (A, f) such that $(x, y) \in \theta_f(x, y)$.
- A cyclic element x' is a colleague of x iff $f^{t_f(x')} = f^{t_f(x)}$.



Preliminaries

Lemma 1

Let $f, g \in A^A$ *be nontrivial operations such that* $\mathrm{Con}(A, f) \subseteq \mathrm{Con}(A, g)$*. Then we have*

1. $\forall x, y \in A : (x, y) \in \alpha \in \mathrm{Con}(A, f) \Longrightarrow (g(x), g(y)) \in \alpha$*, in particular we have* $(g(x), g(y)) \in \theta_f(x, y)$ *and* $\theta_g(x, y) \subseteq \theta_f(x, y)$*.*
2. *Let* B *be a subalgebra of* (A, f)*. Then either* B *is also a subalgebra of* (A, g) *or* g *is constant on* B*, where the constant does not belong to* B*.*

Corollary 2

Let $g_i, i \in I$*, be nontrivial operations on* A*. Then*

$$\mathrm{Con}(A, f) = \bigcap_{i \in I} \mathrm{Con}(A, g_i) \Longleftrightarrow \forall x, y \in A : \theta_f(x, y) = \bigvee_{i \in I} \theta_{g_i}(x, y).$$

Preliminaries

$$\mathrm{Con}(A, f) = \bigcap_{i \in I} \mathrm{Con}(A, g_i) \iff \forall x, y \in A : \theta_f(x, y) = \bigvee_{i \in I} \theta_{g_i}(x, y)$$

How to prove meet-reducibility:
find $g_i, i \in I$, verify that the equation holds

How to prove meet-irreducibility:
prove that there are no such $g_i, i \in I$

Preliminaries

Known:

connected algebras, algebras with small cycles, algebras with short tails

Unknown:

non-connected algebras with at least one cycle with at least 3 elements and there exists element x such that $f(x)$ is nocyclic

Aim

In this talk, we will focus on prime-cycled algebras.

Definition 3

Let (A, f) be a monounary algebra. (A, f) is said to be a **prime-cycled algebra** if each cycle of (A, f) contains a prime number of elements.

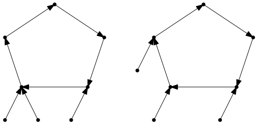
Prime-cycled algebras

Theorem 4

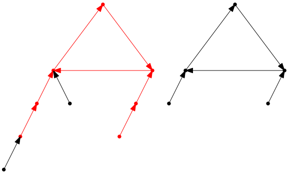
Let (A, f) be a prime-cycled algebra such that each cycle of (A, f) is a p-cycle. $\mathrm{Con}(A, f)$ *is $\wedge$-irreducible in $\mathcal{E}_A$ iff one of the following holds:*

1. (A, f) *is a permutation algebra with* $|A| = 2$, *or*
2. (A, f) *is a permutation algebra with short tails such that* $|A| \geq 3$ *and there are at least two cycles in* (A, f), *or*
3. (A, f) *contains a connected subalgebra B such that there is* $x \in B$ *with* $t_f(x) \geq 2$ *and* $\mathrm{Con}(B, f \restriction B)$ *is $\wedge$-irreducible in* $\mathcal{E}_B$, *or*
4. (A, f) *is non-connected algebra and there are distinct noncyclic elements* $a, b, c, d \in A$ *such that* $f(a), f(c)$ *are cyclic,* $f(b) = a$, $f(d) = c$ *and* $f(a) \neq f(c)$, *or*
5. (A, f) *is non-connected algebra and there are distinct noncyclic elements* $a, b, c, d, e \in A$ *such that* $f(a), f(c)$ *are cyclic,* $f(b) = a$, $f(d) = c$, $f(e) = d$ *and* $f(a) = f(c)$.

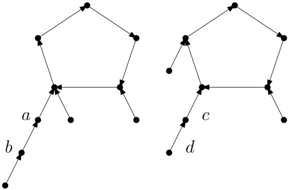
Prime-cycled algebras



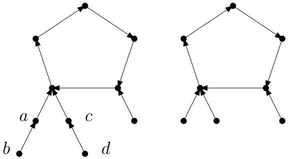
(a) meet-irreducible



(b) meet-irreducible



(c) meet-irreducible

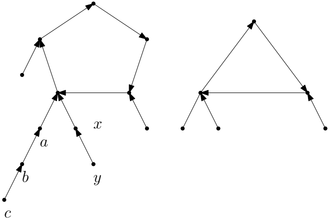


(d) meet-reducible

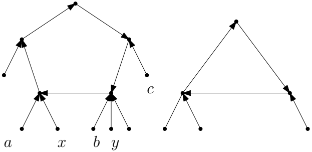
New results

Lemma 5

Let (A, f) be a monounary algebra and $g \in A^A: \ g(x) = f(x')$.
Then for every $x, y \in A: \ \theta_g(x, y) \subseteq \theta_f(x, y)$.



(a) $f(x)$



(b) $g(x)$

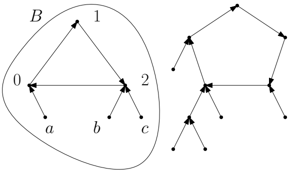
New results

Lemma 6

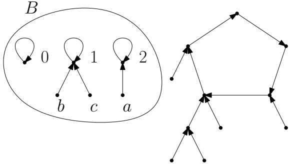
Let (A, f) be a monounary algebra, and let us denote the set of all components of (A, f) with short tails as B. We define the operation $g \in A^A$ as follows:

$$g(x) = \begin{cases} f(x) & \text{if } x \in A \setminus B, \\ x' & \text{if } x \in B. \end{cases}$$

Then for every $x, y \in A : \theta_g(x, y) \subseteq \theta_f(x, y)$.



(a) $f(x)$

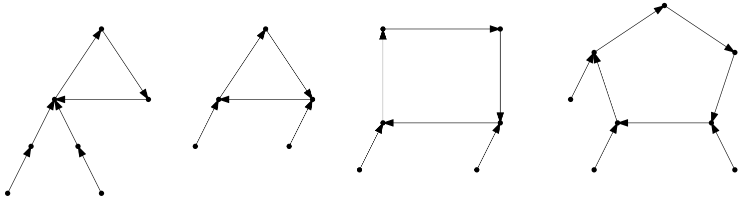


(b) $g(x)$

New results

Proposition 7

Let (A, f) be a monounary algebra such that it contains subalgebras B and $C = A \setminus B$. Let $(B, f \upharpoonright B)$ be a algebra with at least one long tail, such that each cycle of $(B, f \upharpoonright B)$ is a n-cycle, $n \geq 2$ and $t_f(x) \leq 2$ for every $x \in B$. Let $(C, f \upharpoonright C)$ be algebra with short tails. If for every $x, y \in A : t_f(x) = t_f(y) = 2$ implies that $f^2(x) = f^2(y)$, then $\mathrm{Con}(A, f)$ is $\wedge$-reducible in $\mathcal{E}_A$.

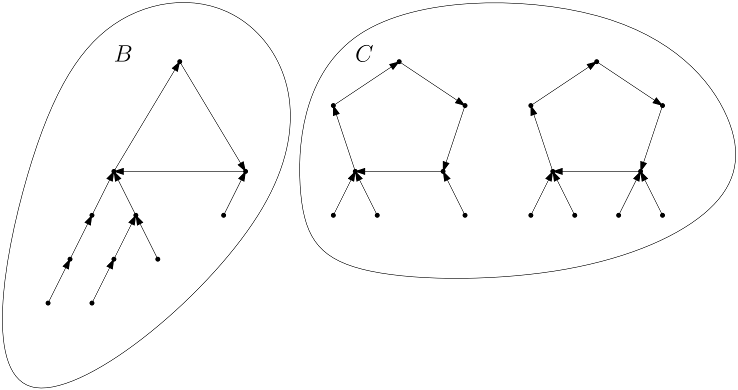


New results

Proposition 8

Let (A, f) *be a monounary algebra such that it contains subalgebras* B *and* $C = A \setminus B$. *Let* $(B, f \restriction B)$ *be a conected algebra with at least one long tail, such that its cycle is a p-cycle,* p *is odd prime and* $t_f(x) \leq p$ *for every* $x \in B$. *Let* $(C, f \restriction C)$ *be algebra with short tails, such that each cycle of* $(C, f \restriction C)$ *has prime length and these lengths are coprime with* p. *If for every* $x, y \in A : t_f(x) = t_f(y) = 2$ *implies that* $f^2(x) = f^2(y)$, *then* $\mathrm{Con}(A, f)$ *is* $\wedge$*-reducible in* $\mathcal{E}_A$.

New results



New results

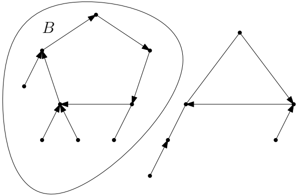
Proposition 9

Let (A, f) *be a algebra with a subalgebra* $(B, f \upharpoonright B)$ *such that* $(B, f \upharpoonright B)$ *is a permutation-algebra with short tails and each cycle containing* $n \geq 2$ *cyclic elements. Let other cycles of* (A, f) *have lengths coprime with* n*. If*

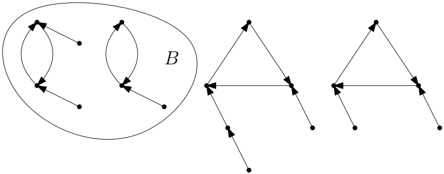
1. $A \setminus B \neq \emptyset$ *and* B *contains exactly one 2-cycle, or*
2. $A \setminus B \neq \emptyset$ *and* $n = p^m$ *for some prime number* p *and* $m \in \mathbb{N}$*, or*
3. $n = 2$ *and* B *contains at least 2 cycles, or*
4. $n \neq p^m$ *for any prime number* p *and* $m \in \mathbb{N}$*,*

then $\mathrm{Con}(A, f)$ *is* $\wedge$*-reducible in* $\mathcal{E}_A$*.*

New results



(a)

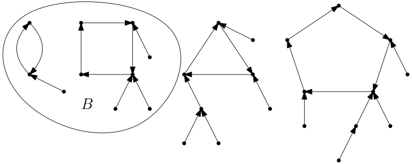


(b)

New results

Proposition 10

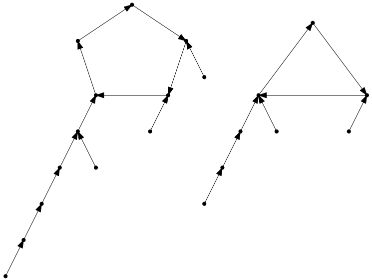
Let (A, f) be a monounary algebra such that B is set of all its components without long tails and $C = A \setminus B$ where $B, C \neq \emptyset$. Let the cycles of B contain $c_1, c_2, ..., c_k$ elements such that $\{c_1, ..., c_k\} \neq \{1\}$ and the cycles of C contain $n_1, n_2, ..., n_m$ elements such that $\{n_1, ..., n_k\} \neq \{1\}$. If $\forall i \in \{1, ..., k\}, \forall j \in \{1, ..., m\} : \gcd(c_i, n_j) = 1$, then $\mathrm{Con}(A, f)$ is $\wedge$-reducible in $\mathcal{E}_A$.



New results

Proposition 11

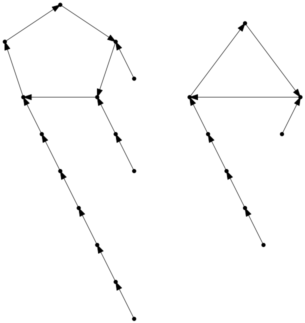
Let (A, f) be a prime-cycled monounary algebra with at least 2 components such that every component has at most one long tail with length at most the number of elements of its cycle. If length of every cycle is different prime number, then $\mathrm{Con}(A, f)$ is $\wedge$-reducible in $\mathcal{E}_A$.



New results

Proposition 12

Let (A, f) *be a prime-cycled mono-nounary algebra with components* $C_1, C_2, ..., C_k, k \in \mathbb{N}$ *such that component* C_i *contains* $p_i \geq 3$ *cyclic elements and* $p_i \neq p_j$ *for* $i \neq j$. *Let for* $\forall i \in \{1, 2, ..., k\}$ *exists an acyclic element* $a_i \in C_i$ *such that* $t_f(a_i) \geq p_i + 1$. *If there exists acyclic element* $b \in A$ *such that* $t_f(b) = 2$ *and for every* i: $f^{t_f(b)}(b) \neq f^{t_f(a_i)}(a_i)$, *then* $\mathrm{Con}(A, f)$ *is* $\wedge$*-irreducible in* $\mathcal{E}_A$.



Sources

1. Jakubíková-Studenovská D., Pöschel R., Radeleczki S., 2018. The lattice of congruence lattices of algebras on a finite set. In: *Algebra Universalis*. Vol. 79(4). ISSN 1420-8911.
2. Jakubíková-Studenovská D., Janičková L., 2018. Meet-irreducible congruence lattices. In: *Algebra Universalis*. Vol. 79(89). ISSN 1420-8911.
3. Jakubíková-Studenovská D., Janičková L., 2020. Congruence lattices of connected monounary algebras. In: *Algebra Universalis*. Vol. 81(54). ISSN 1420-8911.

Thank you for your attention.