A Few Words on Metric Ultraproduct of Groups[1]

Jakub Gismatullin (University of Wrocław)

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Ultraproduct

Let

1. $\mathcal{G} = (G_m)_{m\in\mathbb{N}}$ be a family of first order structures, groups;
2. $\mathcal{U}$ – a non-principal ultrafilter on $\mathbb{N}$.

Definition

Ultraproduct of $\mathcal{G}$ is

$$G^* = \prod_{m\in\mathbb{N}} G_m / \sim$$

where

$$(g_m) \sim (h_m) \iff \{m \in \mathbb{N} : g_m = h_m\} \in \mathcal{U}$$

Ultraproducts are very useful, one can prove compactness theorem, many applications in mathematics (algebra, topology).

Generalization: metric ultraproduct

A *metric group* $(G, \|\cdot\|)$ is a group with a norm $\|\cdot\|\colon G \to [0,\infty)$ such that $\|gh\| \leq \|g\| + \|h\|$, $\|g^{-1}\| = \|g\|$, $\|g\| = 0 \Leftrightarrow g = e$, and $\|g\| = \|hgh^{-1}\|$ (**bi-invariance**), Norm gives a metric $\|\cdot\| \quad \rightsquigarrow \quad d(g,h) = \|gh^{-1}\|$ and vice versa $d(\cdot,\cdot) \quad \rightsquigarrow \quad \|g\| = d(g,e)$

Definition

$\mathcal{G} = (G_m, \|\cdot\|_m)_{m\in\mathbb{N}}$ – a family of metric groups, $\mathcal{U}$ – a non-principal ultrafilter on $\mathbb{N}$. *Metric ultraproduct* is

$$G^*_{\text{met}} = \prod_{m\in\mathbb{N}}^{\text{met}} G_m = G_{\text{fin}}/N_{\mathcal{U}}$$

where

$$G_{\text{fin}} = \left\{(g_m) \in \prod_{m\in\mathbb{N}} G_m : \sup_{m\in\mathbb{N}} \|g_m\|_m < \infty\right\} \text{ oraz } N_{\mathcal{U}} = \left\{(g_m) : \lim_{m\to\mathcal{U}} \|g_m\|_m = 0\right\}$$

(the infinitesimal subgroup $N_{\mathcal{U}}$ is a normal subgroup of G_{fin})

G^*_{met} is a topological space, topology comes from a canonical norm (bi-invariant):

$$\|\cdot\|\colon G^*_{\text{met}} \to \mathbb{R}_{\geq 0} \text{ defined by } \|(g_m)/N_{\mathcal{U}}\| = \lim_{m\to\mathcal{U}} \|g_m\|_m.$$

Examples of norms and metrics

Examples of bounded and unbounded norms

- Discrete norm: $\|g\| := \begin{cases} 1 & : g \neq e \\ 0 & : g = e \end{cases}$
- Hamming norm S_n: $\sigma \in S_n$, $\|\sigma\|_H := \|\{i : \sigma(i) \neq i\}\|$
- Rank norm on $\mathrm{GL}_n(F)$ (F: field) $\|g\|_r := rank(g - I)\,(= \dim(Im(g - I)))$
- Conjugacy length (pseudonorm) on a finite group G:

$$\|g\|_= \ell_c(g) := \frac{\log\left|g^G\right|}{\log|G|}$$

  it is a norm when $Z(G) = \{e\}$
- Invariant word norm of a group G: let $S = S^{-1} \subseteq G$ be a normal subset (that is $s \in S \to s^x = x^{-1}sx \in S$)

$$\|g\|_S = \min\{n : g \text{ is a product of } n \text{ conjugates of elements from } S\}.$$

Motivation: Gottschalk conjecture, ,,1-1" $\Rightarrow$,,onto"

A bit of mathematics: let A be a finite set and G be a group. G acts on

$$A^G = \{f\colon G \to A\}$$

by homeomorphisms

$$g \cdot f(x) = f(g^{-1}x), \text{ for } g, x \in G, \ f \in A^G.$$

This actions is called *Bernoulli shift*.

Definition

A *cellular automaton* is a function $T\colon A^G \to A^G$, which is continuous and G-equivariant $T(g \cdot f) = g \cdot T(f)$.

Conjecture (W. Gottschalk,'72)

If a cellular automaton $T\colon A^G \to A^G$ is ,,1-1" (injective), then T is ,,onto" (surjective).

Theorem (Ax–Grothendieck theorem)

If $P\colon \mathbb{C}^n \to \mathbb{C}^n$ is polynomial mapping and P is ,,1-1", then P is ,,onto".

Proof: Compactness thm. applied to an elementary statement about polynomials over finite field.

Metric ultraproduct and Gottschalk conjecture

Let

$$\mathcal{S} = \prod_{m \in \mathbb{N}}^{\text{met}} \left(S_m, \frac{1}{m} \| \cdot \|_H \right)$$

be a metric ultraproduct of permutation groups S_m with the normalised Hamming norm

Definition

- A group G is sofic, if G is a subgroup of $\mathcal{S}$.
- $\mathcal{S}$ is called an *universal sofic group*.

Theorem (M. Gromov '99)

Gottschalk conjecture is true for sofic groups: If a cellular automaton $T \colon A^{\mathcal{S}} \to A^{\mathcal{S}}$ is ,,1-1", then T is ,,onto".

A major, unresolved problem in group theory: are all finitely generated groups sofic?

Aim

Our aim is to understand (first order metric) logic structure of metric ultraproducts in general

Compactness theorem for ultraproduct

$X \subseteq \mathcal{G}^*$ is *internal* if there is a collection of sets $\{X_n\}_{n \in \mathbb{N}}$, $X_n \subseteq G_n$ such that

$$X = \frac{X_0 \times X_1 \times X_2 \times \cdots}{\sim}.$$

Theorem

$(X_m)_{m \in \mathbb{N}}$ – a sequence of internal subsets of $\mathcal{G}^*$, the following conditions are equivalent:

1. $\mathcal{G}^* = \bigcup_{m \in \mathbb{N}} X_m$
2. *there is $N \in \mathbb{N}$ such that*

$$\mathcal{G}^* = X_0 \cup \ldots \cup X_N$$

Compactness theorem for metric ultraproduct

$X \subseteq \mathcal{G}^*_{\text{met}}$ is *metrically internal* if there is a collection of sets $\{X_n\}_{n\in\mathbb{N}}$, $X_n \subseteq G_n$ such that

$$X = \frac{X_0 \times X_1 \times X_2 \times \cdots}{N_{\mathcal{U}}},$$

where $N_{\mathcal{U}}$ is the infinitesimal subgroup.

Theorem (J. G., K. Majcher, M. Ziegler)

$(X_m)_{m\in\mathbb{N}}$ – a sequence of metrically internal subsets of $\mathcal{G}^*_{met}$, where $X_m = (X_{m,0} \times X_{m,1} \times X_{m,2} \times \ldots)/N_{\mathcal{U}}$, $X_{m,n} \subseteq G_n$, the following conditions are equivalent:

1. $\mathcal{G}^*_{met} = \bigcup_{m\in\mathbb{N}} X_m$
2. *for any sequence $(\varepsilon_0, \varepsilon_1, \ldots) \subset \mathbb{R}_{>0}$ there is $N \in \mathbb{N}$ such that*

$$\mathcal{G}^*_{met} = X_0\mathcal{B}(\varepsilon_0) \cup \ldots \cup X_N\mathcal{B}(\varepsilon_N),$$

where $\mathcal{B}(\varepsilon) = \{g : \|g\| < \varepsilon\}$.

Corollary

$(X_n)_{n\in\mathbb{N}}$ – an increasing sequence of internal subsets of $\mathcal{G}^*_{met}$, If $\mathcal{G}^*_{met} = \bigcup_{m\in\mathbb{N}} X_m$, then there is $N \in \mathbb{N}$ such that

$$\mathcal{G}^*_{met} = X_N \cdot X_N.$$

Application: metric ultraproducts as simple groups

A group G is *simple* is G has no nontrivial proper normal subgroups.

Question

When a (discrete) ultraproduct of groups is simple?

In case of discrete norm, we have the following easy criterion

Fact

$\prod_{m \in \mathbb{N}} G_m / \mathcal{U}$ *is simple* $\Leftrightarrow$ *there is* $N \in \mathbb{N}$*, such that for* $\mathcal{U}$*-almost* $m \in \mathbb{N}$ *group* G_m *is* N*-uniformly simple, that is*

$$G_m = \left(g^{G_m} \cup {g^{-1}}^{G_m} \right)^{\leq N}$$

for all $g \in G_m, g \neq e$*, where* $g^G = \{h^{-1} g h : h \in G\}$.

Examples of simple metric ultraproducts of groups

Question

When a metric ultraproduct of groups is simple?

Some results:

Theorem (G. Elek - E. Szabó, '05)

$\mathcal{S} = \prod_{m \in \mathbb{N}}^{met}(S_m, \frac{1}{m}\|\cdot\|_H)$ *is a simple group*

Let $\{(G_m, \ell_c)\}_{m\in\mathbb{N}}$ be a family of finite simple groups, where $\ell_c(g) = \frac{\log|g^G|}{\log|G|}$.

Theorem (N. Nikolov '09, Stolz - Thom, '14, Ivanov, '14)

1. $\prod_{m \in \mathbb{N}}^{met}(G_m, \ell_c)$ *is a simple group*
2. *Metric ultraproduct of centerless projective classical groups (e.g. PGL) over finite fields is a simple group*

When a metric ultrapower of a group is simple?

$(G, \|\cdot\|)$ - metric group

Theorem (JG, K. Majcher, M. Ziegler)

Metric ultrapower G^*_{met} of G is simple $\Leftrightarrow$ for all $r > 0$ and $t > 0$, for every infinite sequence $(\varepsilon_0, \dots, \varepsilon_n, \dots) \subset \mathbb{R}_{>0}$, there is $N \in \mathbb{N}$ such that for all $g \in G$, $r < \|g\| < t$

$$B_t(e) \subseteq C_0(g)B_{\varepsilon_0}(e) \cup \dots \cup C_N(g)B_{\varepsilon_N}(e).$$

Corollary

$\prod_{m\in\mathbb{N}}^{met}(S_\infty, \frac{1}{m}\|\cdot\|_H)$ *is simple*

Metric ultrapower of matrix groups with ranks norm

K_m - field, $G_m = \mathrm{PSL}_m(K_m)$ (more generally $G_m(K_m)$ – simple centerless Chevalley group $(Z(G_m(K_m)) = \{e\})$)

When each K_m is simple,

$$\prod_{m\in\mathbb{N}}^{\text{met}} (G_m, \ell_c)$$

is simple, where $\ell_c(g) := \frac{\log|g^G|}{\log|G|}$ (Stolz - Thom, '14, Nikolov '09).

How about infinite fields? Consider another norm, e.g. rank norm

$$\|g\|_r := \frac{1}{m} rank(g - I).$$

We conjecture, that for all fields K_m, metric ultrapower $\prod_{m\in\mathbb{N}}^{\text{met}} (G_m(K_m), \|\cdot\|_r)$ is simple.

Corollary

If each field K_m is algebraically closed, then $\prod_{m\in\mathbb{N}}^{met} (G_m(K_m), \|\cdot\|_r)$ is simple.

Idea of the proof: ℓ_c and $\|\cdot\|_r$ are asymptotically equivalent, use Liebeck-Shalev result for the rank norm, which is first order expressible.

An open problem

In every ultraproduct of groups $G^* = \prod_{m\in\mathbb{N}} G_m / \sim$, it is true that if we have an element of arbitrarily large finite order, then there is an element of infinite order:

$$\forall n \in \mathbb{N} \;\; \exists g_n \in G^* \;\; n < o(g_n) < \infty \implies \exists g_\infty \in G^* \;\; o(g_\infty) = \infty.$$

How about metric ultraproduct $\mathcal{G}^*_{\text{met}}$? We can prove, that the same is true, if $\mathcal{G}^*_{\text{met}}$ is nilpotent.

However, this does not seem to be true in general, in the metric ultraproduct $\mathcal{G}^*_{\text{met}}$, but we cannot find a counterexample.

Thank you for your attention!