Amalgamation via interpolation: Beyond the congruence extension property

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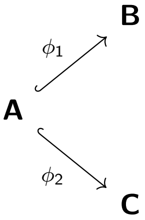
The amalgamation property

The amalgamation property is a fundamental categorical-algebraic property, and it dates back to Schreier's work on group theory.

A class K of algebras has the amalgamation property (or AP) if every span $\langle \phi_1 \colon \mathbf{A} \to \mathbf{B}, \phi_2 \colon \mathbf{A} \to \mathbf{C} \rangle$ of algebras in K can be completed in K so the following diagram commutes:



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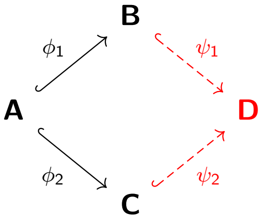


Syntactic interpolation

Algebraic logicians explored amalgamation a lot because it corresponds to syntactic interpolation properties of various logical systems, but less appreciated is the fact that this is an entirely universal-algebraic phenomenon.

Let K be any class of similar algebras. As usual, the equational consequence of K is the relation $\models_{\mathsf{K}}$ between sets of equations and equations defined by

$$\Sigma \models_{\mathsf{K}} \epsilon \iff \bar{\epsilon} \in \mathrm{Cg}_{\mathbf{F}(X)}(\bar{\Sigma}),$$

where X is some set of variables containing all those appearing in $\Sigma \cup \{\epsilon\}$, $\mathbf{F}(X)$ is the free algebra on X, and $\bar{\cdot}$ denotes the projection of the absolutely free algebra on X to $\mathbf{F}(X)$.

The equational interpolation property

Let $\bar{x}$, $\bar{y}$, $\bar{z}$, and so on stand for disjoint collections of variables.

Defn:

A class K has the equational interpolation property if for any $\Sigma \subseteq Eq(\bar{x}, \bar{y})$ and $\epsilon \in Eq(\bar{y}, \bar{z})$ such that $\Sigma \models_{\mathsf{K}} \epsilon$, there exists $\Pi \subseteq Eq(\bar{y})$ such that $\Sigma \models_{\mathsf{K}} \Pi$ and $\Pi \models_{\mathsf{K}} \epsilon$.

Thm (Czelakowski-Pigozzi?):

Let V be a variety.

1. If V has the AP, then V has the equational interpolation property.
2. If additionally V has the CEP, then the equational interpolation property for V and the AP are equivalent.

The gap between AP and interpolation

There are varieties with the equational interpolation property but not the AP, e.g. semigroups.

On the other hand, Kearnes has shown (1989) that for a residually small, congruence-modular variety, the AP implies the CEP.

The point of this work is to isolate a property ??? that is weaker than CEP such such EqIP + ??? holds iff AP does. We exhibit such a property that we call the Robinson extension property.

The Robinson property

Actually, for quasivarieties, AP directly corresponds to a syntactic property called the Robinson property. This holds for Q if for any $\Sigma \subseteq Eq(\bar{x}, \bar{y})$ and $\Pi \cup \{\epsilon\} \subseteq Eq(\bar{y}, \bar{z})$ satisfying

1. $\Sigma \models_{\mathsf{Q}} \delta \iff \Pi \models_{\mathsf{Q}} \delta$ for all $\delta \in Eq(\bar{y})$,
2. $\Sigma \cup \Pi \models_{\mathsf{Q}} \epsilon$,

then we have $\Pi \models_{\mathsf{Q}} \epsilon$.

The Robinson extension property

Let K be any class of similar algebras. We say that K has the Robinson extension property if for any $\Sigma \subseteq Eq(\bar{x}, \bar{y})$ and $\Pi \cup \{\epsilon\} \subseteq Eq(\bar{y}, \bar{z})$ satisfying

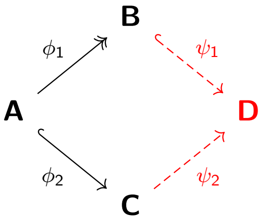
1. $\Sigma \models_{\mathsf{K}} \delta \iff \Pi \models_{\mathsf{K}} \delta$ for all $\delta \in Eq(\bar{y})$,
2. $\Sigma \cup \Pi \models_{\mathsf{K}} \epsilon$,
3. $\Sigma \models_{\mathsf{K}} \rho \implies \Pi \models_{\mathsf{K}} \rho$ for all $\rho \in Eq(\bar{y}, \bar{z})$,

then we have $\Pi \models_{\mathsf{Q}} \epsilon$.

Back to diagrams: the EP

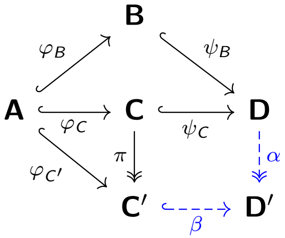
There is a helpful back-and-forth between these kinds of syntactic properties and diagrammatic properties.

For instance, it has long been known that the CEP is equivalent, for quasivarieties, to the so-called extension property:



Back to diagrams: the REP

Let Q be any quasivariety. Then Q has the REP if for each Q-pushout diagram $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \varphi_B, \varphi_C, \psi_B, \psi_C \rangle$, surjective homomorphism $\pi\colon \mathbf{C} \to \mathbf{C}'$, and injective homomorphism $\varphi_{C'}\colon \mathbf{A} \to \mathbf{C}'$ such that the resulting diagram commutes there exist an algebra $\mathbf{D}' \in$ Q, a surjective homomorphism $\alpha\colon \mathbf{D} \to \mathbf{D}'$ and an embedding $\beta\colon \mathbf{C}' \to \mathbf{D}'$, which complete the commuting diagram:



EP $\Rightarrow$ REP, REP + EqIP $\iff$ AP

One can show that the extension property (and therefore the CEP) implies the REP for any quasivariety.

The REP is also strictly weaker than the CEP: For instance, the variety of groups has the REP but not the CEP.

Thm (F., Hortelano-Martín, Metcalfe, and Santschi, 2026+):

Let Q be any quasivariety. Then Q has the AP if and only if Q has the REP and the equational interpolation property.

Note also that the variety of semigroups has EqIP but not REP.

Final remarks

The gap between interpolation and amalgamation is extremely subtle and there are many open problems.

We still don't have any example of a variety that has the REP but lacks both the AP and CEP. It is quite possible that they coincide for certain nice varieties.

However, the interplay between syntactic properties and categorical-algebraic/diagrammatic properties has proven very fruitful and has revealed, e.g., the precise role of pushouts in interpolation.

Thank you!