A syntactic characterization of weakly Mal'tsev varieties

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108. AAA, Vienna

Structure of the talk

1. Internal structures
2. From Mal'tsev to weakly Mal'tsev categories
3. A syntactic characterization of weakly Mal'tsev varieties

1. Internal structures

A **reflexive graph** is given by:

- a set C_0 of objects,
- a set C_1 of morphisms,
- a domain function $d : C_1 \to C_0$,
- a codomain function $c : C_1 \to C_0$,
- an identities function $e : C_0 \to C_1$

such that

$$de(x) = x = ce(x). \qquad \text{(incidence axioms)}$$

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Definition

An **internal reflexive graph** in a category $\mathscr{C}$ is given by a diagram

$$C_1 \underset{c}{\overset{d}{\underset{\longleftarrow e \longrightarrow}{\longrightarrow}}} C_0$$

in $\mathscr{C}$ such that

$$de = 1_{C_1} = ce. \qquad \text{(incidence axioms)}$$

A **(small) category** is given by:

- a reflexive graph (C_0, C_1, d, c, e),
- a multiplication function $m : C_{\to\to} := \{(f, g) \in C_1 \times C_1 \mid c(f) = d(g)\} \to C_1$

such that

$$dm(f, g) = d(f), \qquad cm(f, g) = c(g), \qquad \text{(incidence axioms)}$$
$$m(f, ec(f)) = f = m(de(f), f), \qquad \text{(unitality axioms)}$$
$$m(m(f, g), h) = m(f, m(g, h)). \qquad \text{(associativity axioms)}$$

Definition

An **internal category** in a category $\mathscr{C}$ with pullbacks is given by a diagram

$$C_{\to\to} \xrightarrow{m} C_1 \underset{c}{\overset{d}{\underset{\longleftarrow e \longrightarrow}{\longrightarrow}}} C_0,$$

in $\mathscr{C}$, where $C_{\to\to}$ is given by the pullback

$$\begin{array}{ccc} C_{\to\to} & \xrightarrow{p_2} & C_1 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle d} \\ C_1 & \xrightarrow[c]{} & C_0, \end{array}$$

such that:

- (C_0, C_1, d, c, e) is an internal reflexive graph in $\mathscr{C}$,
- the usual incidence, unitality and associativity axioms for m hold.

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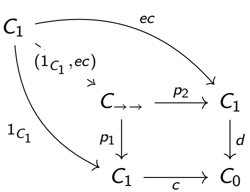
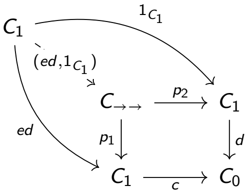
$$m(1_{C_1}, ec) = 1_{C_1} = m(ed, 1_{C_1}) \qquad \text{(unitality axioms)}$$





A (small) category (C_0, C_1, d, c, e, m) is a **(small) groupoid** if there exists an inverses function

$$i : C_1 \to C_1$$

such that

$$di(f) = c(f), \qquad ci(f) = d(f), \qquad \text{(incidence axioms)}$$
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Definition

An internal category (C_0, C_1, d, c, e, m) in a category $\mathscr{C}$ with pullbacks is an **internal groupoid** if there exists a morphism

$$i : C_1 \to C_1$$

such that the usual incidence and inverse axioms for i hold.

$$C_{\to\to} \xrightarrow{\ m\ } C_1 \circlearrowleft^{i} \quad C_1 \underset{c}{\overset{d}{\rightrightarrows}} C_0, \quad C_0 \xrightarrow{\ e\ } C_1$$

2. From Mal'tsev to weakly Mal'tsev categories

Proposition (Mal'tsev; 1954)

Let $\mathbb{V}$ be a variety of universal algebras. TFAE:

1. $\mathbb{V}$ is congruence permutable, i.e., for any algebra $A \in \mathbb{V}$ and $R, S \in \mathrm{Cong}(A)$, it holds that

$$R \circ S = S \circ R.$$

2. There exists a ternary term $p \in \mathsf{F}(x, y, z)$ in the algebraic theory of $\mathbb{V}$ that satisfies the identities

$$p(x, x, y) = y \qquad \text{and} \qquad p(x, y, y) = x.$$

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3. Any reflexive relation in $\mathbb{V}$ is a congruence.
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Examples of congruence permutable varieties

- Grp ($p(x, y, z) = x \cdot y^{-1} \cdot z$),

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Definition (Carboni, Pedicchio, Pirovano; 1992)

A **Mal'tsev category** is a finitely complete category $\mathscr{C}$ in which any of the following equivalent conditions hold:

1. Any reflexive relation in $\mathscr{C}$ is a congruence.
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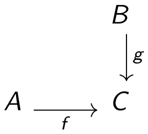
- Any abelian category: $\mathrm{Sh}(\mathrm{Ab})$
- The dual of any elementary topos: $\mathrm{Set}^{\mathrm{op}}$

Proposition (Bourn; 1996)

Let $\mathscr{C}$ be a finitely complete category. TFAE:

1. $\mathscr{C}$ is a Mal'tsev category.
2. Given split epimorphisms f, g in $\mathscr{C}$ with respective splittings r, s, i.e., $fr = 1_C = gs$, then, in the pullback of f and g, the induced pullback injections e_1, e_2 are jointly extremally epimorphic,



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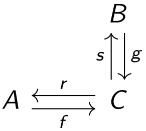

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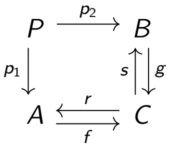

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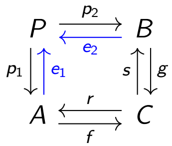

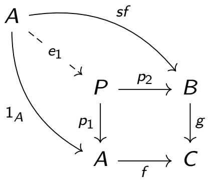

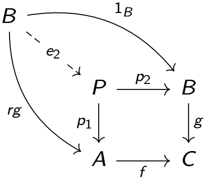

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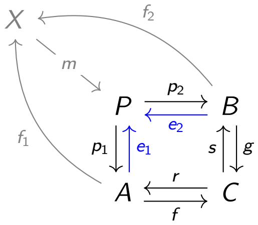

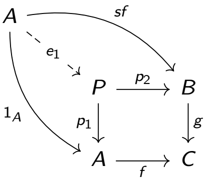

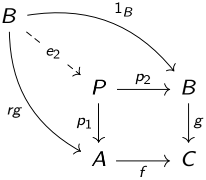

Proposition

Any reflexive graph in a Mal'tsev category $\mathscr{C}$ allows at most one internal category structure.

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Proof

Let

$$C_1 \underset{c}{\overset{d}{\underset{\longleftarrow e \longrightarrow}{\longrightarrow}}} C_0$$

be a reflexive graph in $\mathscr{C}$, i.e. d, c are split epimorphisms with common splitting e.

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Hence the pullback injections e_1, e_2 in

$$\begin{array}{ccc} C_{\to\to} & \underset{e_2}{\overset{p_2}{\rightleftarrows}} & C_1 \\ {\scriptstyle p_1}\downarrow\uparrow{\scriptstyle e_1} & & {\scriptstyle e}\uparrow\downarrow{\scriptstyle d} \\ C_1 & \underset{c}{\overset{e}{\leftrightarrows}} & C_0 \end{array}$$

are jointly extremally epimorphic.

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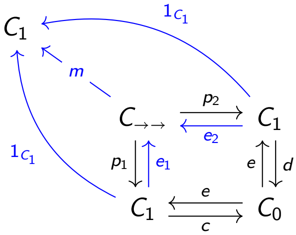

are jointly extremally epimorphic.
The unitality axioms are exactly $me_1 = 1_{C_1}$ and $me_2 = 1_{C_1}$.

Remark

- In a Mal'tsev category, a reflexive graph as above is an internal category if and only the kernel congruences of d, c centralize each other.

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- If $\mathscr{C}$ is a congruence permutable variety with Mal'tsev term $p(x, y, z)$, and $[\mathrm{Eq}(d), \mathrm{Eq}(c)] = \Delta_{C_1}$, then, for any $f, g \in C_1$ with $c(f) = d(g)$,

$$m(f, g) = p(f, ec(f), g).$$

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Remark

If $\mathscr{C}$ is congruence permutable variety with Mal'tsev term $p(x, y, z)$, then the inverse morphism $i : C_1 \to C_1$ for an internal category is given by

$$i(f) = p(ed(f), f, ec(f)).$$

There are categories such as $\mathrm{DistrLatt}$, $\mathrm{CancCommMon}$, $\mathrm{Top}^{\mathrm{op}}$ in which:

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- There are internal categories that are **not** internal groupoids.

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Definition (Martins-Ferreira; 2008)

A **weakly Mal'tsev category** is a category that has all pullbacks of split epimorphisms along split epimorphisms and in which the pullback injections e_1, e_2 in a pullback

$$\begin{array}{ccc} P & \overset{p_2}{\underset{e_2}{\rightleftarrows}} & B \\ {\scriptstyle p_1}\downarrow\uparrow{\scriptstyle e_1} & & {\scriptstyle s}\uparrow\downarrow{\scriptstyle g} \\ A & \overset{r}{\underset{f}{\leftrightarrows}} & C \end{array}$$

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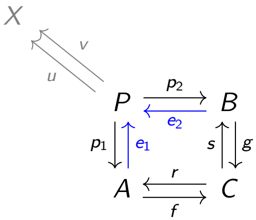

as above are jointly epimorphic, i.e., if $ue_1 = ve_1$ and $ue_2 = ve_2$, then $u = v$, Proposition

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Remark

In the weakly Mal'tsev category $\mathrm{CancCommMon}$, the diagram

$$C_1 := \{(n,m) \in \mathbb{N}_0 \times \mathbb{N}_0 \mid n \leq m\} \underset{c(n,m)=m}{\overset{d(n,m)=n}{\rightleftarrows}} \mathbb{N}_0 =: C_0 \quad (e(n)=(n,n) : \mathbb{N}_0 \to C_1)$$

yields an internal category which is not an internal groupoid.

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Proposition (Janelidze, Martins-Ferreira; 2012)

Let $\mathscr{C}$ be a finitely complete category. TFAE:

1. $\mathscr{C}$ is a weakly Mal'tsev category.
2. Any reflexive strong relation is a congruence.
3. Any reflexive strong relation is transitive.
4. Any reflexive strong relation is symmetric.

3. A syntactic characterization of weakly Mal'tsev varieties

Proposition

Let $\mathbb{V}$ be a variety of universal algebras. TFAE:

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3. For any pullback

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as above in $\mathbb{V}$, the pullback injections e_1, e_2 are jointly extremally epimorphic.

Proof of "3. $\Rightarrow$ 2."

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⇔ The pullback injections e_1, e_2 in the pullback

$$\begin{array}{ccc} P & \underset{e_2}{\overset{p_2}{\rightleftarrows}} & \mathrm{F}(x,y) \\ {\scriptstyle p_1}\downarrow\uparrow{\scriptstyle e_1} & & {\scriptstyle s}\uparrow\downarrow{\scriptstyle f} \\ \mathrm{F}(x,y) & \underset{f}{\overset{r}{\leftrightarrows}} & \mathrm{F}(x) \end{array}$$

are jointly extremally epimorphic, where f, r, s are the unique morphisms such that $f(x) = x = f(y)$, $r(x) = x$ and $s(x) = y$, respectively.

Proof of "3. ⇒ 2."

For any pullback

$$\begin{array}{ccc} P & \underset{e_2}{\overset{p_2}{\rightleftarrows}} & B \\ {\scriptstyle p_1}\downarrow\uparrow{\scriptstyle e_1} & & {\scriptstyle s}\uparrow\downarrow{\scriptstyle g} \\ A & \underset{f}{\overset{r}{\leftrightarrows}} & C \end{array}$$

as above in $\mathbb{V}$, the pullback injections e_1, e_2 are jointly extremally epimorphic.

$\Leftrightarrow$ The pullback injections e_1, e_2 in the pullback

$$\begin{array}{ccc} P & \underset{e_2}{\overset{p_2}{\rightleftarrows}} & \mathsf{F}(x,y) \\ {\scriptstyle p_1}\downarrow\uparrow{\scriptstyle e_1} & & {\scriptstyle s}\uparrow\downarrow{\scriptstyle f} \\ \mathsf{F}(x,y) & \underset{f}{\overset{r}{\leftrightarrows}} & \mathsf{F}(x) \end{array}$$

are jointly extremally epimorphic, where f, r, s are the unique morphisms such that $f(x) = x = f(y)$, $r(x) = x$ and $s(x) = y$, respectively.

$\Leftrightarrow$ The induced morphism $[e_1, e_2] : \mathsf{F}(x,y) + \mathsf{F}(x,y) \to P$ is a surjective homomorphism.

Proof of "3. $\Rightarrow$ 2."

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$\Leftrightarrow$ The induced morphism $[e_1, e_2] : \mathsf{F}(x,y) + \mathsf{F}(x,y) \to P$ is a surjective homomorphism.

$\Leftrightarrow$ The element $(y, x) \in P = \{(t, t') \in \mathsf{F}(x,y) \times \mathsf{F}(x,y) \mid t(x,x) = t'(x,x)\}$ lies in $\mathrm{Im}([e_1, e_2])$.

Lemma

It holds that

$$\mathrm{Im}([e_1, e_2]) = \{(p(x, x, y), p(x, y, y)) \mid p \in \mathsf{F}(x, y, z)\}.$$

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"$\subseteq$": The elements $(x,x) = e_2(x)$, $(x,y) = e_1(x) = e_2(y)$ and $(y,y) = e_1(y)$ lie in $\mathrm{Im}([e_1, e_2])$.

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$$p((x, x), (x, y), (y, y)) = (p(x, x, y), p(x, y, y))$$

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"$\supseteq$": Use that $\mathsf{F}(x,y) + \mathsf{F}(x,y)$ can be described as a quotient of $\mathsf{T}(\mathsf{F}(x,y) \dot\cup \mathsf{F}(x,y))$ ...

Strategy for weakly Mal'tsev varieties

$\mathbb{V}$ is a weakly Mal'tsev category.

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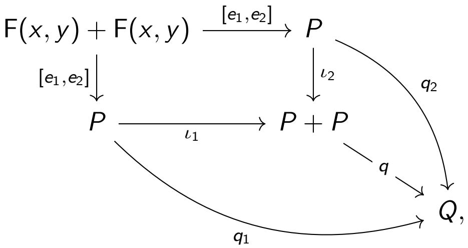
$\Leftrightarrow$ It holds that $$q_1(y,x) = q_2(y,x).$$

- We construct q_1, q_2 by means of the commutative diagram



where $q : P + P \to Q$ is the coequalizer of $\iota_1[e_1, e_2], \iota_2[e_1, e_2]$.

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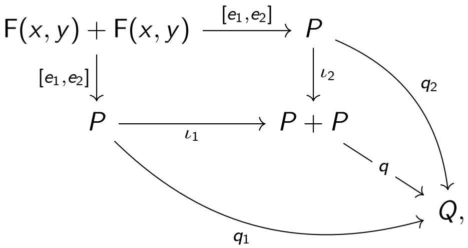

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This means that we describe Q as the quotient of $P + P$ by the congruence C which is generated by the subset

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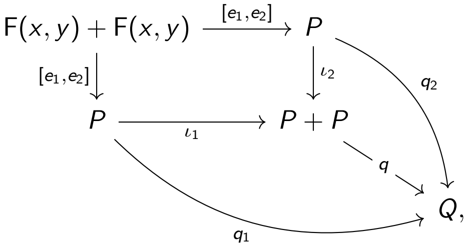

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- Thanks to the above lemma, the congruence C is generated by the subset

$$S := \{(\iota_1(p(x,x,y), p(x,y,y)), \iota_2(p(x,x,y), p(x,y,y))) \mid p \in \mathsf{F}(x,y,z)\}.$$

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- We construct C by taking
  1. the reflexive closure,
  2. the symmetric closure,
  3. the closure under operations,
  4. the transitive closure of S.
- Using the description of $P + P$ as a quotient of $\mathrm{F}(P \dot\cup P)$, the condition $(\iota_1(y,x), \iota_2(y,x)) \in C$ implies the existence of terms as in:

Theorem (E., Jacqmin, Martins-Ferreira; 2024)

Let $\mathbb{V}$ be a variety of universal algebras. TFAE:

1. $\mathbb{V}$ is a weakly Mal'tsev category.
2. In the algebraic theory of $\mathbb{V}$, there exist
   - $k, m, N \in \mathbb{N}_0$,
   - binary terms

     $$f_1, g_1, \dots, f_k, g_k,$$

   - ternary terms

     $$p_1, \dots, p_m,$$

   - $(1+k+m)$-ary terms

     $$\eta_1^{(1)}, \eta_2^{(1)}, \varepsilon_1^{(1)}, \varepsilon_2^{(1)}, \dots, \eta_1^{(N+1)}, \eta_2^{(N+1)}, \varepsilon_1^{(N+1)}, \varepsilon_2^{(N+1)},$$

   - $(2(1+k+m)+2)$-ary terms

     $$\sigma_1, \dots, \sigma_{N+1},$$

   - $2(1+k+2m)$-ary terms

     $$s_1 \dots, s_N$$

   satisfying the identities

$$u = \sigma_1(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \eta_1^{(1)}(u, \vec{v}, \vec{w}), \eta_2^{(1)}(u', \vec{v}', \vec{w}')),$$

$$\sigma_i(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \varepsilon_1^{(i)}(u, \vec{v}, \vec{w}), \varepsilon_2^{(i)}(u', \vec{v}'\ \vec{w}')) = s_i(u, \vec{v}, \vec{w}, \vec{w}, u', \vec{v}', \vec{w}', \vec{w}'),$$

$$s_i(u, \vec{v}, \vec{w}, \vec{w}', u', \vec{v}', \vec{w}', \vec{w}) = \sigma_{i+1}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \eta_1^{(i)}(u, \vec{v}, \vec{w}), \eta_2^{(i)}(u', \vec{v}'\ \vec{w}')),$$

$$\sigma_{N+1}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \varepsilon_1^{(N+1)}(u, \vec{v}, \vec{w}), \varepsilon_2^{(N+1)(u'.\vec{v}',\vec{w}')}) = u',$$

where $\vec{v} = (v_1, \dots, v_k)$ and $\vec{w} = (w_1, \dots, w_m)$ and analogously for $\vec{v}'$ and $\vec{w}'$;

- $$
\begin{gathered}
u=\sigma_1(u,\vec{v},\vec{w},u',\vec{v}',\vec{w}',\eta_1^{(1)}(u,\vec{v},\vec{w}),\eta_2^{(1)}(u',\vec{v}',\vec{w}')),\\
\sigma_i(u,\vec{v},\vec{w},u',\vec{v}',\vec{w}',\varepsilon_1^{(i)}(u,\vec{v},\vec{w}),\varepsilon_2^{(i)}(u',\vec{v}'\,\vec{w}'))=s_i(u,\vec{v},\vec{w},\vec{w},u',\vec{v}',\vec{w}',\vec{w}'),\\
s_i(u,\vec{v},\vec{w},\vec{w}',u',\vec{v}',\vec{w}',\vec{w})=\sigma_{i+1}(u,\vec{v},\vec{w},u',\vec{v}',\vec{w}',\eta_1^{(i)}(u,\vec{v},\vec{w}),\eta_2^{(i)}(u',\vec{v}'\,\vec{w}')),\\
\sigma_{N+1}(u,\vec{v},\vec{w},u',\vec{v}',\vec{w}',\varepsilon_1^{(N+1)}(u,\vec{v},\vec{w}),\varepsilon_2^{(N+1)(u'.\vec{v}',\vec{w}')})=u',
\end{gathered}
$$

  where $\vec{v}=(v_1,\dots,v_k)$ and $\vec{w}=(w_1,\dots,w_m)$ and analogously for $\vec{v}'$ and $\vec{w}'$;

- $$
\begin{aligned}
&\eta_\alpha^{(i)}(y,f_1(x,y),\dots,f_k(x,y),p_1(x,x,y),\dots,p_m(x,x,y))\\
=&\varepsilon_\alpha^{(i)}(y,f_1(x,y),\dots,f_k(x,y),p_1(x,x,y),\dots,p_m(x,x,y)),
\end{aligned}
$$

  $$
\begin{aligned}
&\eta_\alpha^{(i)}(x,g_1(x,y),\dots,g_k(x,y),p_1(x,y,y),\dots,p_m(x,y,y))\\
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\end{aligned}
$$

  for all $\alpha\in\{1,2\}$ and $i\in\{1,\dots,N+1\}$;

- $$u = \sigma_1(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \eta_1^{(1)}(u, \vec{v}, \vec{w}), \eta_2^{(1)}(u', \vec{v}', \vec{w}')),$$
$$\sigma_i(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \varepsilon_1^{(i)}(u, \vec{v}, \vec{w}), \varepsilon_2^{(i)}(u', \vec{v}'\ \vec{w}')) = s_i(u, \vec{v}, \vec{w}, \vec{w}, u', \vec{v}', \vec{w}', \vec{w}'),$$
$$s_i(u, \vec{v}, \vec{w}, \vec{w}', u', \vec{v}', \vec{w}', \vec{w}) = \sigma_{i+1}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \eta_1^{(i)}(u, \vec{v}, \vec{w}), \eta_2^{(i)}(u', \vec{v}'\ \vec{w}')),$$
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where $\vec{v} = (v_1, \ldots, v_k)$ and $\vec{w} = (w_1, \ldots, w_m)$ and analogously for $\vec{v}'$ and $\vec{w}'$;
- $$\begin{aligned}&\eta_\alpha^{(i)}(y, f_1(x, y), \ldots, f_k(x, y), p_1(x, x, y), \ldots, p_m(x, x, y))\\ =&\varepsilon_\alpha^{(i)}(y, f_1(x, y), \ldots, f_k(x, y), p_1(x, x, y), \ldots, p_m(x, x, y)),\end{aligned}$$
$$\begin{aligned}&\eta_\alpha^{(i)}(x, g_1(x, y), \ldots, g_k(x, y), p_1(x, y, y), \ldots, p_m(x, y, y))\\ =&\varepsilon_\alpha^{(i)}(x, g_1(x, y), \ldots, g_k(x, y), p_1(x, y, y), \ldots, p_m(x, y, y))\end{aligned}$$
for all $\alpha \in \{1, 2\}$ and $i \in \{1, \ldots, N+1\}$;
- $$f_i(x, x) = g_i(x, x)$$
for all $i \in \{1, \ldots, k\}$.

Example of congruence permutable varieties

If $\mathbb{V}$ is a congruence permutable variety with Mal'tsev term $p \in \mathsf{F}(x, y, z)$ satisfying $p(x, x, y) = y$ and $p(x, y, y) = y$, then

- k can be chosen to be 0: no binary terms necessary,
- m can be chosen to be 1: $p_1(x, y, z) = p(x, y, z)$,
- N can be chosen to be 1: all the terms $\eta_\alpha^{(i)}, \varepsilon_\alpha^{(i)}, \sigma_i, s_j$ can be chosen to be projections.

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Example of varieties of distributive lattices

If $\mathbb{V}$ is a variety of distributive lattices, then

- k can be chosen to be 0: no binary terms necessary,
- m can be chosen to be 3: $p_1(x, y, z) = x$, $p_2(x, y, z) = y$, $p_3(x, y, z) = z$,
- N can be chosen to be 5.

Theorem (E., Jacqmin, Martins-Ferreira; 2024)

Let $\mathbb{V}$ be a variety of universal algebras. TFAE:

1. Any reflexive regular relation in $\mathbb{V}$ is symmetric.
2. There are terms as in the above theorem that satisfy the same identities except

$$f_i(x,x) = g_i(x,x)$$

for all $i \in \{1, \ldots, k\}$.

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Remark (Martins-Ferreira, Van der Linden; 2014, Martins-Ferreira, Rodelo, Van der Linden; 2014)

- In a finitely complete weakly Mal'tsev category, any internal category is an internal groupoid if and only if any preorder is an equivalence relation.
- The same is true for regular categories.
- For a variety $\mathbb{V}$ of universal algebras, these conditions are further equivalent to $\mathbb{V}$ being n-permutable for some $n \in \mathbb{N}$.
- There are categories that are weakly Mal'tsev and Goursat (=3-permutable) but not Mal'tsev.

Given terms as in the above theorem, one can show that $q_1(y,x) = q_2(y,x)$ by

- inserting for

$$\begin{aligned}
&u : q_1(y,x),\\
&\vec{v} : (q_1(f_1,g_1),\dots,q_1(f_k,g_k)),\\
&\vec{w} : (q_1(p_1(x,x,y),p_1(x,y,y)),\dots,q_1(p_m(x,x,y),p_m(x,y,y))),\\
&u' : q_2(y,x),\\
&\vec{v}' : (q_2(f_1,g_1),\dots,q_2(f_k,g_k)),\\
&\vec{w}' : (q_2(p_1(x,x,y),p_1(x,y,y)),\dots,q_2(p_m(x,x,y),p_m(x,y,y))),
\end{aligned}$$

- using that

$$q_1(p_i(x,x,y),p_i(x,y,y)) = q_2(p_i(x,x,y),p_i(x,y,y))$$

  for all $i \in \{1,\dots,m\}$.

Proof

- The subset
$$\{(p(x,x,y),p(x,y,y)) \mid p \in \mathrm{F(x,y,z)}\}$$
is not only $\mathrm{Im}([\mathrm{e}_1,\mathrm{e}_2])$ but also the smallest reflexive relation
$$R \rightarrowtail \mathrm{F}(x,y) \times \mathrm{F}(x,y)$$
on $\mathrm{F}(x,y,)$ that contains (x,y).
- $\mathbb{V}$ is a Mal'tsev category if and only if $(y,x) \in R$.
- Any reflexive regular relation in $\mathbb{V}$ is symmetric if and only if the smallest reflexive regular relation R' on $\mathrm{F}(x,y)$ that contains (x,y) also contains (y,x).
- The relation R' can be described as the equalizer of the cokernel pair $q'_1, q'_2 : \mathrm{F}(x,y) \times \mathrm{F}(x,y) \to Q'$ of $R' \rightarrowtail \mathrm{F}(x,y) \times \mathrm{F}(x,y)$, whose elements are those $(t,t') \in \mathrm{F}(x,y) \times \mathrm{F}(x,y)$ such that $q'_1(t) = q'_2(t')$. This means that $(y,x) \in R'$ if and only if $q'_1(y,x) = q'_2(y,x)$.