A new representation of finite Hoops using a new type of product of structures

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Motivation

- Goal: Introduce an f-product for hoops with strong associativity behaviour.
- Main representation (finite case): $\mathbf{A} \cong \mathbf{F} \ltimes \mathbf{A}/F$ for any filter $F \in \mathit{Fil}\,(\mathbf{A})$.
- Consequence: every finite hoop is, in this sense, a product of finite MV-chains.
- Inspiration: wreath / semidirect products and the Krohn–Rhodes paradigm.

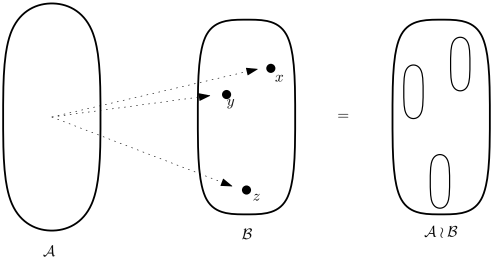

Figure 1: Idea of the f-product.

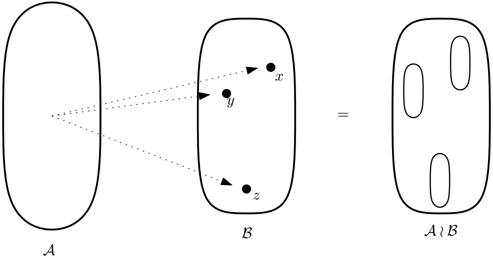

Figure 1: Idea of the f-product.

If $\mathbf{A}$ and $\mathbf{B} =$ are hoops and let $(T_b)_{b \in |\mathbf{B}|}$ is system of some transformations then

$$|\mathcal{A} \ltimes \mathcal{B}| = \sum_{x \in |\mathbf{B}|} T_b(\mathbf{A})$$

A wreat product

- The idea of associativity is based on the fact that the relevant transformations are rightly composed (as a mapping).
- An important example of a "right definition" is the wreath product, and the Krohn-Rhode theorem is the model I based my work on.

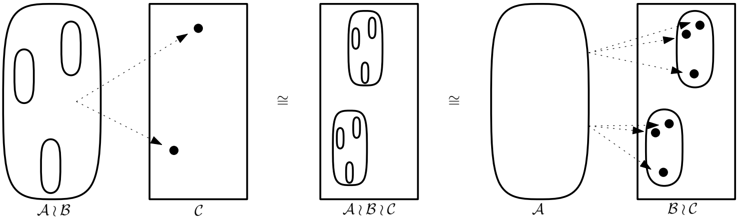

Figure 2: Associativity of a wreath product.

Hoops

Definition

A hoop is the algebra $\mathbf{A} = (A; \cdot, \rightarrow, 1)$ *of the type* $\langle 2, 2, 0 \rangle$*, where* $(A; \cdot, 1)$ *is a commutative monoid and satisfying the identities*

(H1) $x \rightarrow x = 1$,

(H2) $(x \cdot y) \rightarrow z = x \rightarrow (y \rightarrow z)$,

(H3) $x \cdot (x \rightarrow y) = y \cdot (y \rightarrow x)$.

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The induced order is given by

$$x \leq y \text{ if and only if } 1 = x \rightarrow y$$

and moreover

$$x \wedge y = x \cdot (x \rightarrow y).$$

Alternativelly it can be defined as naturally ordered commutative semigroup satisfying an adjointness property

$$x \cdot y \leq z \Longleftrightarrow x \leq y \rightarrow z.$$

Frame Title

If $\mathbf{A} = (A; \cdot, \to, 1)$ is a hoop then a filter is nonempty set $F \subseteq A$

(F1) if $x \in F$ and $x \leq y$ then $y \in F$,

(F2) if $x, y \in F$ then $x \cdot y \in F$.

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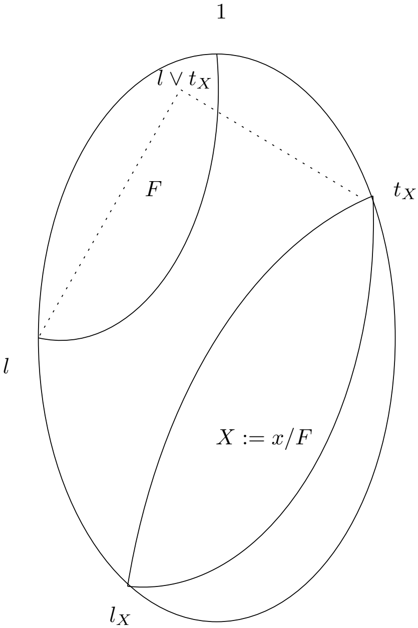
The set of all filters of the hoop $\mathbf{A}$ is denoted by $\mathbf{Fil}\,\mathbf{A}$. It is well known that any filter F induce the congruence

$$\theta_F = \{\langle x, y \rangle \in A^2 \mid (x \rightarrow y) \cdot (y \rightarrow x) \in F\}$$

The main idea



There exists the nucleus $\gamma_X\colon F \longrightarrow F$ (closure operator satisying $\gamma_X(a)\cdot\gamma_X(b) \leq \gamma_X(a\cdot b)$) defined by $\gamma_X(a) = t_X \to (t_X \cdot a)$. Then there is bijection $t_X \cdot \_\colon \gamma_X F \longrightarrow X$. Then

$$A \cong \sum_{X\in \mathbf{A}/F} \gamma_X F.$$

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$$A \cong \sum_{X \in \mathbf{A}/F} \gamma_X F.$$

But also it satisfies that

$$\gamma_X F \cong (I \vee t_X] := \{a \in F \colon a \leq I \vee t_t\}.$$

Hence,

$$A \cong \sum_{X \in \mathbf{A}/F} \gamma_X F \cong \sum_{X \in \mathbf{A}/F} (I \vee t_X].$$

A new product of hoops

Definition

If $\mathbf{A} = (A; \cdot, \rightarrow, 1)$ *and* $\mathbf{B} = (B; \cdot, \rightarrow, 1)$ *are hoops. Then the mapping* $f : A \longrightarrow B$ *we call a product morphism from the hoop* $\mathbf{A}$ *to the hoop* $\mathbf{B}$ *if it satisfies*

(pM1) $f(1) = 1$,

(pM2) $f(x) \cdot f(y) = f(x \cdot y) = f(x) \wedge f(y) = f(x \wedge y)$.

for any $x, y \in A$*.*

A new product of hoops

Theorem

If $\mathbf{A} = (A; \cdot, \rightarrow), 1$ *and* $\mathbf{B} = (B; \cdot \rightarrow, 1)$ *are hoops and* $f : B \longrightarrow A$ *is a product morphism, then the algebra*

$$\mathbf{A} \ltimes_f \mathbf{B} = (\sum_{x \in B}(f(x)], \cdot, \rightarrow, (1,1))$$

such that

($\cdot$) $(a, x) \cdot (b, y) := (a \cdot b, x \cdot y)$,

($\rightarrow$) $(a, x) \rightarrow (b, y) := (f(x \rightarrow y) \wedge (a \rightarrow b), x \rightarrow y)$

for any $(a, x), (b, y) \in \sum_{x \in B}(f(x)]$ *is a hoop. We say that* $\mathbf{A} \ltimes_f \mathbf{B}$ *is a* f*-product of the hoops* $\mathbf{A}$ *and* $\mathbf{B}$

Examples

Example

If $\mathbf{A}$ *and* $\mathbf{B}$ *are arbitrary hoops and* $\varepsilon\colon \mathbf{B} \longrightarrow \mathbf{A}$ *is the constant mapping* $\varepsilon(x) = 1$ *then clearly* ε *is a product morphism and it is the greatest morphism with respect to the natural order. It is easy to check that*

$$\mathbf{A} \ltimes_{\varepsilon} \mathbf{B} = \mathbf{A} \times \mathbf{B}.$$

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Example

If **A** *and* **B** *are arbitrary hoops such that there exists the least element* $0 \in \mathbf{A}$*. Then there is the product morphism* $\sigma\colon \mathbf{B} \longrightarrow \mathbf{A}$ *defined by*

$$\sigma(x) = \begin{cases} 1 & \textit{iff} \quad x = 1 \\ 0 & \textit{iff} \quad x \neq 1 \end{cases}$$

then clearly σ *is the least morphism with respect to the natural order and it is easy to check that*

$$\mathbf{A} \ltimes_{\sigma} \mathbf{B} = \mathbf{B} \oplus \mathbf{A}.$$

An associativity of product

Definition

If $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{C}$ *are hoops then we say that* (f, g) *is a left associated pair of product morphims with respect to* $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ *if* $f \colon \mathbf{B} \to \mathbf{A}$ *and*

$$g \colon \mathbf{C} \longrightarrow \mathbf{A} \ltimes_f \mathbf{B}$$

are product morphisms. (We therefore have the product $(\mathbf{A} \ltimes_f \mathbf{B}) \ltimes_g \mathbf{C}$*.)*

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are product morphisms. (We therefore have the product $(\mathbf{A} \ltimes_f \mathbf{B}) \ltimes_g \mathbf{C}$*.)*
Similarly, (f, g) *is a right associated pair of product morphims with respect to* $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ *if* $g : \mathbf{C} \to \mathbf{B}$ *and*

$$f : \mathbf{B} \ltimes_g \mathbf{C} \longrightarrow \mathbf{A}$$

are product morphisms. (We therefore have the product $\mathbf{A} \ltimes_f (B \ltimes_g \mathbf{C})$*.)*

Associativity

If (f, g) is left associated pair of product morphisms with respect to $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ and if we denote the mapping $g \colon \mathbf{C} \longrightarrow \mathbf{A} \ltimes_f \mathbf{B}$ by $g(c) = (g_1(c), g_2(c))$ then we can define the mappings:

- $\overline{g} \colon \mathbf{C} \longrightarrow \mathbf{B}$ by $\overline{g}(c) = g_2(c)$ for all $c \in C$,
- $\overline{f} \colon \mathbf{B} \ltimes_{\overline{g}} \mathbf{C} \longrightarrow \mathbf{A}$ by $\overline{f}(b, c) = f(b) \wedge g_1(c)$ for any $(b, c) \in |\mathbf{B} \ltimes_{\overline{g}} \mathbf{C}|$

such that $\alpha(f, g) = (\overline{f}, \overline{g})$ is a right associated pair of product morphims with respect to $(\mathbf{A}, \mathbf{B}, \mathbf{C})$.

Associativity

If (f, g) is right associated pair of product morphisms with respect to the triple of hoops $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ then we can define the mappings:

- $\overline{f}\colon \mathbf{B} \longrightarrow \mathbf{A}$ by $\overline{f}(b) = f(b, 1)$ for any $b \in B$,
- $\overline{g}\colon \mathbf{C} \longrightarrow \mathbf{A} \ltimes_{\overline{f}} \mathbf{B}$ by $\overline{g}(c) = (f(g(c), c), g(c))$ for any $c \in C$,

such that $\beta(f, g) = (\overline{f}, \overline{g})$ is left associated pair of product morphisms with respect to $(\mathbf{A}, \mathbf{B}, \mathbf{C})$.

Associtivity

The corespondences α and β between left and right associated pairs of product morphisms with respect to $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ are mutually inverse bijective mappings and if $\alpha(f, g) = (\overline{f}, \overline{g})$ (or equivalently $(f, g) = \beta(\overline{f}, \overline{g})$) then

$$(\mathbf{A} \ltimes_f \mathbf{B}) \ltimes_g \mathbf{C} \cong \mathbf{A} \ltimes_{\overline{f}} (\mathbf{B} \ltimes_{\overline{g}} \mathbf{C}).$$

Decomposition of finite hoops

Lemma

If $\mathbf{A}$ is a finite hoop and $F \in \mathbf{Fil}\,\mathbf{A}$. We denote t_X the top element and l_X the least element of the class X for any $X \in \mathbf{A}/F$. For the simplicity we denote l (instead of l_F) the least element of the filter F. Then it satisfies

(i) $X \leq Y$ *if and only if* $t_X \leq t_Y$,

(ii) $t_X \bullet t_Y \leq t_{X \bullet Y}$ *for any operation* $\bullet$ *belonging to the set* $\{\vee, \wedge, \cdot, \rightarrow\}$,

(iii) $t_X \wedge t_Y = t_{X \wedge Y}$,

(iv) $t_X \rightarrow t_Y = t_{X \rightarrow Y}$,

(v) $t_X \cdot t_{X \rightarrow Y} = t_{X \wedge Y}$,

(vi) $a \rightarrow t_X = t_X$ *and* $t_X \cdot a = t_X \wedge a$ *for any* $a \in F$,

(vii) $t_X \cdot l = t_X \wedge l = l_X$.

for any $X, Y \in \mathbf{A}/F$.

The decomposition theorem

Theorem

If $\mathbf{A}$ *is a finite hoop and* $F \in \mathbf{Fil}\,\mathbf{A}$ *then the mapping*

$$\psi\colon \mathbf{A}/F \longrightarrow \mathbf{F}$$

defined by $\psi(X) = t_X \vee I$ *is a product morphism and moreover it satisfies*

$$\mathbf{A} \cong \mathbf{F} \ltimes_\psi (\mathbf{A}/F).$$

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Definition

The hoop $\mathbf{A}$ *is irreducible if* $\mathbf{A} \cong \mathbf{B} \ltimes_f \mathbf{C}$ *implies that* $\mathbf{B}$ *or* $\mathbf{C}$ *is a trivial hoop (for arbitrary product morphism* $f\colon \mathbf{C} \longrightarrow \mathbf{B}$*).*

Theorem

The finite hoop is irreducible if and only if it is simple.

The decomposition of finite hoops

Theorem (Blok, W. J. and Ferreirim, I. M. A.)

Simple finite hoops are just finite MV-chains.

Theorem

(i) For any finite hoop $\mathbf{A}$ *there exists a (finite) sequences of finite MV-chains* $(\mathbf{M}_{i_1}, \mathbf{M}_{i_2}, \ldots, \mathbf{M}_{i_n})$ *and appropriate product morphisms* $(f_1, \ldots, f_{n-1})$ *such that*

$$\mathbf{A} \cong \mathbf{M}_{i_1} \ltimes_{f_1} (\mathbf{M}_{i_2} \ltimes_{f_2} (\mathbf{M}_{i_3} \cdots \ltimes_{f_{n-2}} (\mathbf{M}_{i_{n-1}} \ltimes_{f_{n-1}} \mathbf{M}_{i_n}) \ldots)).$$

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$$\mathbf{A} \cong (((\mathbf{M}_{i_1} \ltimes_{g_1} \mathbf{M}_{i_2}) \ltimes_{g_2} \mathbf{M}_{i_3}) \cdots \ltimes_{g_{n-2}} \mathbf{M}_{i_{n-1}}) \ltimes_{g_{n-1}} \mathbf{M}_{i_n}.$$

The decomposition of finite hoops

Theorem

If **A** *is an arbitrary finite hoop then the set* **Id** (**A**) *is equivalent to the set of all product morphisms from* **A** *to any finite MV-chain and also to the set of all product morphisms from any finite MV-chain to* **A**.

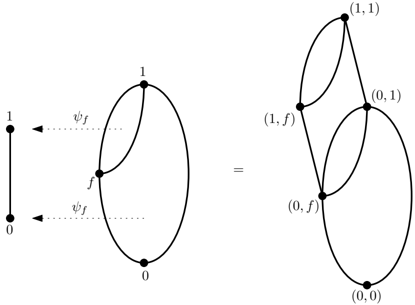

Figure 3: The visualisation of $\mathbf{M} \ltimes_{\psi_f} \mathbf{A}$.

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The decomposition of finite hoops

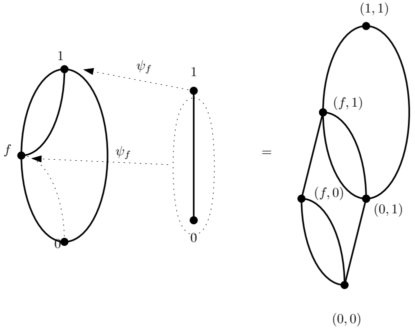

Figure 4: The visualisation of $\mathbf{A} \ltimes_{\psi_f} \mathbf{M}$.

Thank you for your attention!