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Minimal clones generated by $\{0,1\}$-valued majority operations on up to six-element sets

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Clones

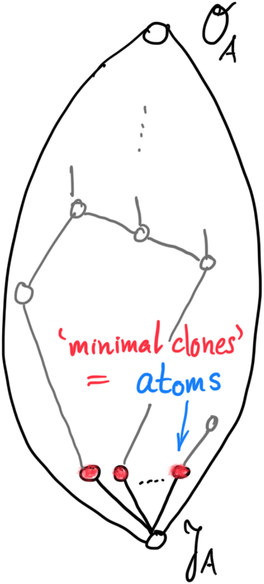
- operations $f\colon A^n \longrightarrow A$, $n \in \mathbb{N}$
- $\mathrm{O}_A = \{ f\colon A^n \longrightarrow A \mid n \in \mathbb{N}_+ \}$
- composition $f \circ (g_1, \dots, g_n)(\boldsymbol{x}) = f(g_1(\boldsymbol{x}), \dots, g_n(\boldsymbol{x}))$
  for each $\boldsymbol{x} = (x_1, \dots, x_m) \in A^m$
- projections $e_i^{(n)}\colon A^n \longrightarrow A \quad (x_1, \dots, x_n) \mapsto x_i$ for $1 \leq i \leq n$
- set of projections $\mathrm{J}_A = \left\{ e_i^{(n)} \,\middle|\, 1 \leq i \leq n < \omega \right\}$.

Clone: a set $F \subseteq \mathrm{O}_A$
with $\mathrm{J}_A \subseteq F$ and F closed under composition

$\mathcal{L}_A = \{ F \subseteq \mathrm{O}_A \mid F \text{ a clone} \}$

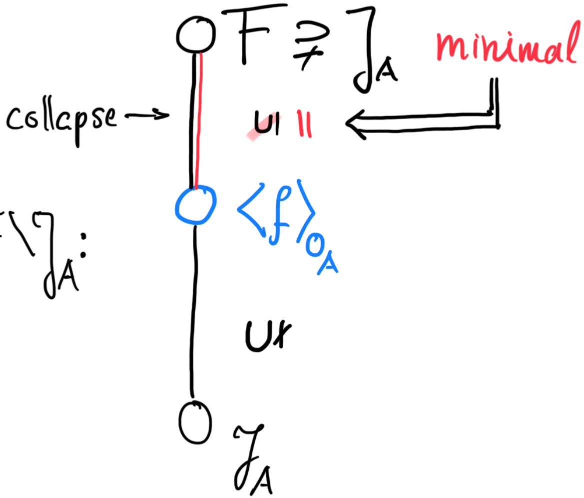
- a complete algebraic lattice
- atomic if A is finite

The clone lattice $\mathcal{L}_A$ for finite A



Minimal clones: singly generated

$\forall\, f \in F \setminus J_A$:

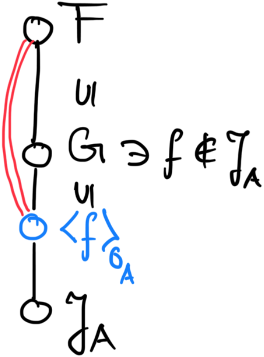


Minimal clones: characterisation

$F \supsetneq J_A \qquad \forall f \in F \setminus J_A: \quad \langle f \rangle_{O_A} = F$

$\forall J_A \subseteq G \subsetneq F:$

collapse $\longrightarrow$



Minimal clones: basic criterion

$F \subseteq \mathrm{O}_A$ is minimal iff

a. $F \supsetneq \mathrm{J}_A$
b. $\forall g \in F \setminus \mathrm{J}_A\colon \quad F = \langle \{g\} \rangle_{\mathrm{O}_A}$

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For $f \in \mathrm{O}_A \setminus \mathrm{J}_A$

$\langle\{f\}\rangle_{\mathrm{O}_A}$ is a minimal clone $\iff \forall g \in \langle\{f\}\rangle_{\mathrm{O}_A} \setminus \mathrm{J}_A\colon \quad f \in \langle\{g\}\rangle_{\mathrm{O}_A}$.

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$f \in \mathrm{O}_A$ is a minimal function $\iff$

- $\langle\{f\}\rangle_{\mathrm{O}_A}$ is minimal $\quad (\implies f \notin \mathrm{J}_A)$
- f has minimum arity among generators of $\langle\{f\}\rangle_{\mathrm{O}_A}$

Rosenberg's Classification Theorem

Necessary condition:

Theorem (Ivo Rosenberg, 1986)

Let $2 \leq |A| < \aleph_0$, $n \in \mathbb{N}$.
If $f\colon A^n \longrightarrow A$ minimal, then one of the following cases holds:

- $n = 1$ and $f \circ f = f$ or $\exists p$ prime: $f^p = \mathrm{id}_A$
- $n = 3$ and $f(x, y, z) \approx x - y + z$ w.r.t. a Boolean group $\langle A; +\rangle$
- $|A| \geq n \geq 2$ and f is a semiprojection
- $\boxed{n = 3 \text{ and } f \text{ is a } \textbf{majority operation}}$

$f\colon A^3 \longrightarrow A$ is a majority operation $\iff$ $f \in \mathrm{Maj}_A$ $\iff$

$$f(x, x, y) \approx f(x, y, x) \approx f(y, x, x) \approx x.$$

S-valued majority operations

Let $S \subseteq A$.

$f\colon A^3 \longrightarrow A$ is an S-valued majority operation $\iff$ $f \in \mathrm{Maj}_A^S$ $\iff$

- $f \in \mathrm{Maj}_A$
- $\forall x, y, z \in A$: $\quad |\{x, y, z\}| = 3 \implies f(x, y, z) \in S$

Subsequently: $S = \{0, 1\}$

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Facts about $f \in \mathrm{Maj}_A$:

- $\langle\{f\}\rangle_{\mathrm{O}_A} \setminus \mathrm{J}_A$ contains only near-unanimity operations
- $\langle\{f\}\rangle_{\mathrm{O}_A}^{(3)} \setminus \mathrm{J}_A \subseteq \mathrm{Maj}_A$
- $f \in \mathrm{Maj}_A^S \implies \langle\{f\}\rangle_{\mathrm{O}_A}^{(3)} \setminus \mathrm{J}_A \subseteq \mathrm{Maj}_A^S$

Machida's Lemma

Lemma (Machida, 2024)

Let $A \supseteq \{0,1\}$. If $f \in \mathrm{Maj}_A^{\{0,1\}}$, then

$$g(x,y,z) :\approx f(f(x,y,z), f(y,z,x), f(z,x,y))$$

is cyclically symmetric, in fact, $g \in \mathrm{cMaj}_A^{\{0,1\}}$.

$\mathrm{cMaj}_A^{\{0,1\}}$

set of cyclically symmetric $\{0,1\}$-valued majority operations

$f\colon A^n \longrightarrow A$ for $n \geq 1$ is cyclically symmetric $\iff$

$$f(x_1, \ldots, x_n) \approx f(x_2, \ldots, x_n, x_1)$$

Consequence of Machida's Lemma

Corollary

Let $f \in \mathrm{Maj}_A^{\{0,1\}}$.

$\langle\{f\}\rangle_{\mathrm{O}_A}$ minimal $\implies \exists g \in \langle\{f\}\rangle_{\mathrm{O}_A} \cap \mathrm{cMaj}_A^{\{0,1\}} \colon \langle\{f\}\rangle_{\mathrm{O}_A} = \langle\{g\}\rangle_{\mathrm{O}_A}$

Search for minimal $\{0,1\}$-valued majority clones
can focus on cyclically symmetric ones!

Minimality criterion for majority clones

General:

Basic criterion: for $f \in \mathrm{O}_A \setminus \mathrm{J}_A$

$\langle\{f\}\rangle_{\mathrm{O}_A}$ is a minimal clone $\iff$ $\forall g \in \langle\{f\}\rangle_{\mathrm{O}_A} \setminus \mathrm{J}_A\colon \quad f \in \langle\{g\}\rangle_{\mathrm{O}_A}$.

For majority:

Simpler criterion (Csákány, Waldhauser): for $f \in \mathrm{Maj}_A$

$\langle\{f\}\rangle_{\mathrm{O}_A}$ is a minimal clone $\iff$ $\forall g \in \langle\{f\}\rangle_{\mathrm{O}_A}^{(3)} \setminus \mathrm{J}_A\colon \quad f \in \langle\{g\}\rangle_{\mathrm{O}_A}^{(3)}$.

Minimality of $f \in \mathrm{Maj}_A$ depends only on (abstract) composition structure of $\langle\{f\}\rangle_{\mathrm{O}_A}^{(3)}$

(3-place Menger algebra)

Our contributions

Goal

Description of all minimal $\langle\{f\}\rangle_{\mathrm{O}_A}$ with $f \in \mathrm{Maj}_A^{\{0,1\}}$

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Description of all minimal $\langle\{f\}\rangle_{O_A}$ with $f \in \cancel{\mathrm{Maj}_A^{\{0,1\}}}\,\mathrm{cMaj}_A^{\{0,1\}}$ (Machida's Lemma)

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Description of all minimal $\langle\{f\}\rangle_{O_A}$ with $f \in \cancel{\mathrm{Maj}_A^{\{0,1\}}}\,\mathrm{cMaj}_A^{\{0,1\}}$

$$|A| = k \implies \left|\mathrm{cMaj}_A^{\{0,1\}}\right| = 2^{\frac{1}{3}k(k-1)(k-2)}$$

$k = 5: \quad 2^{20}$

$k = 6: \quad 2^{40}$

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Description of all minimal $\langle\{f\}\rangle_{\mathrm{O}_A}$ with $f \in \cancel{\mathrm{Maj}_A^{\{0,1\}}}\,\mathrm{cMaj}_A^{\{0,1\}}$

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BVW 2025

$k = 5$ solved! Functions classified up to $\left|\langle\{f\}\rangle_{\mathrm{O}_A}^{(3)}\right|$

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Description of all minimal $\langle\{f\}\rangle_{\mathrm{O}_A}$ with $f \in \cancel{\mathrm{Maj}_A^{\{0,1\}}}\,\mathrm{cMaj}_A^{\{0,1\}}$

$$|A| = k \implies \left|\mathrm{cMaj}_A^{\{0,1\}}\right| = 2^{\frac{1}{3}k(k-1)(k-2)}$$

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$k = 6 : \quad 2^{40}$

BVW 2025

$k = 5$ solved! Functions classified up to $\left|\langle\{f\}\rangle_{\mathrm{O}_A}^{(3)}\right|$

BVW 2026

$k = 6$ solved! Functions classified up to $\left|\langle\{f\}\rangle_{\mathrm{O}_A}^{(3)}\right|$

+ Connections with $k = 5$; classification up to $\left[\langle\{f\}\rangle_{\mathrm{O}_A}^{(3)}\right]_{\cong}$ (isomorphism)

Equivalences on functions

Conjugacy: for $f, g \in \mathrm{O}_A^{(n)}$

$$f \sim g \iff \langle A; g\rangle \cong \langle A; f\rangle$$
$$\iff \exists \varphi \in \operatorname{Sym} A\colon f(x_1,\dots,x_n) \approx \varphi^{-1}(g(\varphi(x_1),\dots,\varphi(x_n)))$$

Note: $f \sim g \implies \langle\{f\}\rangle_{\mathrm{O}_A} \sim \langle\{g\}\rangle_{\mathrm{O}_A}$ (action isomorphic, conjugate)

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Reversed function of $f \in \mathrm{O}_A^{(n)}$

$$\bar{f}(x_1,\dots,x_n) :\approx f(x_n,\dots,x_1)$$

Note: $\langle\{f\}\rangle_{\mathrm{O}_A} = \langle\{\bar{f}\}\rangle_{\mathrm{O}_A}$

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Equivalence—conjugacy up to reversal: for $f, g \in \mathrm{O}_A^{(n)}$

$$f \equiv g \iff f \sim g \text{ or } f \sim \bar{g}$$

Note: $f \equiv g \implies \langle\{f\}\rangle_{\mathrm{O}_A} \sim \langle\{g\}\rangle_{\mathrm{O}_A}$ ($\rightsquigarrow$ preserving minimality)

Results 2025

Theorem (BVW 2025) for $A = \{0,\ldots,4\}$

There are subsets $H_1, H_8, H_{16}, H_{64} \subseteq \mathrm{cMaj}_A^{\{0,1\}}$ such that:

- (a) $|H_1| = 8 \quad |H_8| = 15$
  $|H_{16}| = 1 \quad |H_{64}| = 2$
- (b) $\forall f \in H := H_1 \cup H_8 \cup H_{16} \cup H_{64}: \quad f$ is minimal
- (c) $\forall j \in \{1,8,16,64\}\, \forall f \in H_j: \quad \left|\langle\{f\}\rangle^{(3)}_{\mathrm{O}_A}\right| = j + 3$
- (d) $\forall$ minimal $g \in \mathrm{cMaj}_A^{\{0,1\}}\ \exists! f \in H: \quad g \equiv f$
- (e) $\forall$ minimal $h \in \mathrm{Maj}_A^{\{0,1\}}\ \exists f \in H: \quad \langle\{h\}\rangle_{\mathrm{O}_A} \sim \langle\{f\}\rangle_{\mathrm{O}_A}$

Proof.

By computation! (a), (b), (c) also by pen & paper proofs. $\square$

Results 2026

Theorem (BVW 2026) for $A = \{0, \dots, 5\}$

There are subsets $F_1, F_8, F_{16}, F_{64} \subseteq \mathrm{cMaj}_A^{\{0,1\}}$ such that:

- (a) $|F_1| = 43 \quad |F_8| = 139$
  $|F_{16}| = 20 \quad |F_{64}| = 56$
- (b) $\forall f \in F := F_1 \cup F_8 \cup F_{16} \cup F_{64}: \quad f$ is minimal
- (c) $\forall j \in \{1, 8, 16, 64\}\, \forall f \in F_j: \quad \left|\langle\{f\}\rangle_{\mathrm{O}_A}^{(3)}\right| = j + 3$
- (d) $\forall$ minimal $g \in \mathrm{cMaj}_A^{\{0,1\}}\ \exists! f \in F: \quad g \equiv f$
- (e) $\forall$ minimal $h \in \mathrm{Maj}_A^{\{0,1\}}\ \exists f \in F: \quad \langle\{h\}\rangle_{\mathrm{O}_A} \sim \langle\{f\}\rangle_{\mathrm{O}_A}$

Proof.

By computation! For (b) also reduction to $|A| = 5$. $\square$

Clone homomorphisms

Restriction homomorphism

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$\varphi\colon \langle U; h\rangle \longrightarrow \langle V; g\rangle$ iso. $\iff g = h^{(\varphi)} := \varphi \circ h \circ (\varphi^{-1} \times \cdots \times \varphi^{-1})$

In this case $-^{(\varphi)}\colon \langle\{h\}\rangle_{\mathrm{O}_U} \longrightarrow \langle\{g\}\rangle_{\mathrm{O}_V} \qquad \hbar \mapsto \hbar^{(\varphi)}$ clone iso.

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Given $f \in \mathrm{Pol}_A\{U\}$ + an iso $\varphi \colon \langle U; f|_U \rangle \longrightarrow \langle V; g \rangle$ i.e., $g = (f|_U)^{(\varphi)}$

$\Longrightarrow \ r \colon \langle \{f\} \rangle_{\mathrm{O}_A} \longrightarrow \langle \{g\} \rangle_{\mathrm{O}_V} \qquad \hbar \mapsto (\hbar|_U)^{(\varphi)}$ surj. clone hom.

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$\implies r\colon \langle\{f\}\rangle_{\mathrm{O}_A} \longrightarrow \langle\{g\}\rangle_{\mathrm{O}_V} \qquad \hbar \mapsto (\hbar|_U)^{(\varphi)}$ surj. clone hom.

Additionally given $n \in \mathbb{N}$ where $\left|\langle\{f\}\rangle^{(n)}_{\mathrm{O}_A}\right| = \left|\langle\{g\}\rangle^{(n)}_{\mathrm{O}_V}\right| < \aleph_0$

$\implies r_n = r|_{\dots}\colon \langle\{f\}\rangle^{(n)}_{\mathrm{O}_A} \longrightarrow \langle\{g\}\rangle^{(n)}_{\mathrm{O}_V}$ is a Menger algebra iso.

Proof of minimality for $|A| = 6$

For $e \geq 2$:
$\varphi_e\colon U_e \longrightarrow V := \{0, \ldots, 4\}$

$U_e := \{0, 1, \ldots, 5\} \setminus \{e\}$

x	0	1	…	$e-1$	$\cancel{e}$	$e+1$	…	5
$\varphi_e(x)$	0	1	…	$e-1$		e	…	4

For each $j \in \{1, 8, 16, 64\}$ and every $f \in F_j$ on $A = \{0, \ldots, 5\}$

$\exists e \geq 2\, \exists h \in H_j$:

- $(f|_{U_e})^{(\varphi_e)} \equiv h$ (conjugate up to reversal)

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- $\exists \pi \in \operatorname{Sym} V$: $(f|_{U_e})^{(\pi\circ\varphi_e)} = g \in \{h, \bar{h}\}$

$\Longrightarrow r \colon \langle\{f\}\rangle_{O_A} \longrightarrow \langle\{g\}\rangle_{O_V}$ $\quad \hbar \mapsto (\hbar|_{U_e})^{(\pi\circ\varphi_e)}$ clone hom.

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  $\Longrightarrow \; r \colon \langle \{f\} \rangle_{O_A} \longrightarrow \langle \{g\} \rangle_{O_V}$ $\quad \hbar \mapsto (\hbar|_{U_e})^{(\pi \circ \varphi_e)}$ clone hom.
- $\langle \{h\} \rangle_{O_V} = \langle \{\bar{h}\} \rangle_{O_V} = \langle \{g\} \rangle_{O_V}$

Proof of minimality for $|A| = 6$

For $e \geq 2$:
$\varphi_e\colon U_e \longrightarrow V := \{0,\dots,4\}$

$U_e := \{0,1,\dots,5\}\setminus\{e\}$

$$\begin{array}{c|cccccccc} x & 0 & 1 & \dots & e-1 & \cancel{e} & e+1 & \dots & 5 \\ \varphi_e(x) & 0 & 1 & \dots & e-1 & & e & \dots & 4 \end{array}$$

For each $j \in \{1,8,16,64\}$ and every $f \in F_j$ on $A = \{0,\dots,5\}$

$\exists e \geq 2\, \exists h \in H_j$:

- $(f|_{U_e})^{(\varphi_e)} \equiv h$ (conjugate up to reversal)
- $(f|_{U_e})^{(\varphi_e)} \sim g \in \{h, \bar{h}\}$ (conjugate)
- $\exists \pi \in \operatorname{Sym} V$: $(f|_{U_e})^{(\pi\circ\varphi_e)} = g \in \{h,\bar{h}\}$
  $\implies r\colon \langle\{f\}\rangle_{O_A} \longrightarrow \langle\{g\}\rangle_{O_V}$ $\quad \hbar \mapsto (\hbar|_{U_e})^{(\pi\circ\varphi_e)}$ clone hom.
- $\langle\{h\}\rangle_{O_V} = \langle\{\bar{h}\}\rangle_{O_V} = \langle\{g\}\rangle_{O_V}$
- $\left|\langle\{f\}\rangle^{(3)}_{O_A}\right| = j+3 = \left|\langle\{h\}\rangle^{(3)}_{O_V}\right| = \left|\langle\{g\}\rangle^{(3)}_{O_V}\right|$ (as $f \in F_j$, $h \in H_j$)
  $\implies r_3\colon \langle\{f\}\rangle^{(3)}_{O_A} \longrightarrow \langle\{g\}\rangle^{(3)}_{O_V} = \langle\{h\}\rangle^{(3)}_{O_V}$ Menger algebra iso.

Connection between 2025 and 2026

Theorem (BVW 2026)

For $A = \{0, \dots, 5\}$ each $j \in \{1, 8, 16, 64\}$ and $f \in F_j$ there is $h \in H_j$ on $V = \{0, \dots, 4\}$ such that

$$\langle\{f\}\rangle^{(3)}_{\mathrm{O}_A} \cong \langle\{h\}\rangle^{(3)}_{\mathrm{O}_V}$$

and both clones are minimal (by the theorem of 2025).

Classification up to Menger algebra isomorphism

Theorem (BVW 2026)

For minimal $f \in \mathrm{cMaj}_A^{\{0,1\}}$ for $5 \leq |A| \leq 6$,
the cardinality of $\langle \{f\} \rangle_{\mathrm{O}_A}^{(3)}$ determines its isomorphism class.

Proof

Check of $A = \{0, \ldots, 4\}$ is sufficient; $|A| = 6$ follows from prev. thm.

- $f \in H_1$: trivially isomorphic Menger algebras (to $\langle m_0 \rangle_{\mathrm{O}_{\{0,1,2\}}}^{(3)}$)
- $f \in H_8$: $\langle \{f\} \rangle_{\mathrm{O}_A}^{(3)} \cong \langle m_{255} \rangle_{\mathrm{O}_{\{0,1,2\}}}^{(3)}$ (Csákány, 1986)
- $f \in H_{16}$: $|H_{16}| = 1$
- $f, \hat{f} \in H_{64}$: $\langle \{f\} \rangle_{\mathrm{O}_A}^{(3)} \cong \left\langle \{\hat{f}\} \right\rangle_{\mathrm{O}_A}^{(3)}$ (non-trivial isomorphism)

Classification up to Menger algebra isomorphism

Theorem (BVW 2026)

For minimal $f \in \mathrm{cMaj}_A^{\{0,1\}}$ for $2 \leq |A| \leq 6$,
the cardinality of $\langle \{f\} \rangle^{(3)}_{\mathrm{O}_A}$ determines its isomorphism class.

Proof

Check of $A = \{0, \dots, 4\}$ is sufficient; $|A| = 6$ follows from prev. thm.

- $f \in H_1$: trivially isomorphic Menger algebras (to $\langle m_0 \rangle^{(3)}_{\mathrm{O}_{\{0,1,2\}}}$)
- $f \in H_8$: $\langle \{f\} \rangle^{(3)}_{\mathrm{O}_A} \cong \langle m_{255} \rangle^{(3)}_{\mathrm{O}_{\{0,1,2\}}}$ (Csákány, 1986)
- $f \in H_{16}$: $|H_{16}| = 1$
- $f, \hat{f} \in H_{64}$: $\langle \{f\} \rangle^{(3)}_{\mathrm{O}_A} \cong \left\langle \{\hat{f}\} \right\rangle^{(3)}_{\mathrm{O}_A}$ (non-trivial isomorphism)

For $2 \leq |A| \leq 4$: only $\left[\langle m_0 \rangle^{(3)}_{\mathrm{O}_{\{0,1,2\}}}\right]_{\cong}$ and $\left[\langle m_{255} \rangle^{(3)}_{\mathrm{O}_{\{0,1,2\}}}\right]_{\cong}$ appear.

Final words

Thank you for your attention!

Enjoy the rest of the conference.