Generating higher commutators in algebras with a Mal'cev term

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A definition of commutators for Mal'cev algebras

Two-term description of the commutator

Theorem. Let $\mathbf{A}$ be an algebra with Mal'cev term, α_1, α_2 congruences. Then

$$[\alpha_1, \alpha_2] = \Theta_{\mathbf{A}}\Big(\Big\{\big(s(b_1, b_2), t(b_1, b_2)\big) \mid (a_1, b_1) \in \alpha_1, (a_2, b_2) \in \alpha_2, \\ s, t \in \mathrm{Pol}_2(\mathbf{A}), s = t \text{ on } \{(a_1, a_2), (a_1, b_2), (b_1, a_2)\}\Big\}\Big).$$

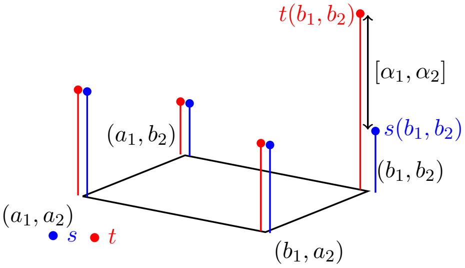

Figure: Commutators as forks.

Initial TikZ draft generated by GPT-5; manually revised by the author.

Two-term description of the commutator

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$$[\alpha_1, \alpha_2] = \Theta_{\mathbf{A}}\Big(\Big\{\big(s(b_1, b_2), t(b_1, b_2)\big) \mid (a_1, b_1) \in \alpha_1, (a_2, b_2) \in \alpha_2, \\ s, t \in \mathrm{Pol}_2(\mathbf{A}), s = t \text{ on } \{(a_1, a_2), (a_1, b_2), (b_1, a_2)\}\Big\}\Big).$$

- Two-term description of commutators (with $s \equiv_{[\alpha_1, \alpha_2]} t$ instead of $s = t$): E. W. Kiss, Three remarks on the modular commutator, 1992 and A. Moorhead, Higher commutator theory for congruence modular varieties, 2018.
- Can be proved from Corollary 6.10 of E. Aichinger, N. Mudrinski, Some applications of higher commutators in Mal'cev algebras, 2010.

Two-term description of the higher commutator

Theorem. Let $\mathbf{A}$ be an algebra with Mal'cev term, $\alpha_1, \dots, \alpha_n$ congruences. Then

$$[\alpha_1, \dots, \alpha_n] = \Theta_{\mathbf{A}}\Big(\Big\{\big(s(b_1, \dots, b_n), t(b_1, \dots, b_n)\big) \mid (a_1, b_1) \in \alpha_1, \dots, (a_n, b_n) \in \alpha_n,$$
$$s, t \in \mathrm{Pol}_n(\mathbf{A}), s = t \text{ on } \big(\bigtimes_{i=1}^{n} \{a_i, b_i\}\big) \setminus \{(b_1, \dots, b_n)\},$$
$$a_1 \neq b_1, \dots, a_n \neq b_n\Big\}\Big).$$

- Use Corollary 6.10 of E. Aichinger, N. Mudrinski, Some applications of higher commutators in Mal'cev algebras, 2010.
- Or: Use Proposition 3.8 of J. Opršal, A relational description of higher commutators in Mal'cev varieties, 2016.

Congruence generation in Mal'cev algebras

Congruence generation in Mal'cev algebras

Theorem. $\mathbf{A}$ algebra with Mal'cev term. Then

$$\Theta_{\mathbf{A}}(a,b) = \{(u(a,\boldsymbol{e}), u(b,\boldsymbol{e})) \mid m \in \mathbb{N}, \boldsymbol{e} \in A^m, u \in \mathrm{Clo}_{m+1}(\mathbf{A})\}.$$

Proof: Compute the diagonal subalgebra of $\mathbf{A} \times \mathbf{A}$ generated by (a,b).

Two-term congruence generation in Mal'cev algebras

Theorem. $\mathbf{A}$ algebra with Mal'cev term. Then

$$\Theta_{\mathbf{A}}(a,b) = \{(s(a,b,\boldsymbol{e}), t(a,b,\boldsymbol{e})) \mid m \in \mathbb{N}, \boldsymbol{e} \in A^m, s,t \in \mathrm{Clo}_{m+2}(\mathbf{A}),$$
$$s(x,x,\boldsymbol{z}) = t(x,x,\boldsymbol{z}) \text{ for all } x \in A, \boldsymbol{z} \in A^m\}.$$

Proof: "$\subseteq$":

- Let $(u^{\mathbf{A}}(a,\boldsymbol{e}), u^{\mathbf{A}}(b,\boldsymbol{e})) \in \Theta_{\mathbf{A}}(a,b)$.
- Let M be the Mal'cev term, and set

$$\begin{aligned} s(x,y,z_0,\boldsymbol{z}) &:= u(M(x,y,z_0),\boldsymbol{z}) \\ t(x,y,z_0,\boldsymbol{z}) &:= u(z_0,\boldsymbol{z}). \end{aligned}$$

- Then $s(x,x,z_0,\boldsymbol{z}) \approx t(x,x,z_0,\boldsymbol{z})$ and
$s^{\mathbf{A}}(a,b,b,\boldsymbol{e}) = u^{\mathbf{A}}(M^{\mathbf{A}}(a,b,b),\boldsymbol{e}) = u^{\mathbf{A}}(a,\boldsymbol{e}), \quad t^{\mathbf{A}}(a,b,b,\boldsymbol{e}) = u^{\mathbf{A}}(b,\boldsymbol{e})$.

Congruence generation using consensus oriented terms

Theorem. Let V be a variety with a Mal'cev term M, let $\mathbf{A} \in V$, $a, b \in A$. Then

$$\Theta_{\mathbf{A}}(a,b) = \{(s(a,b,\boldsymbol{e}), t(a,b,\boldsymbol{e})) \mid m \in \mathbb{N}, \boldsymbol{e} \in A^m, s,t \text{ are terms}, \\ V \models s(x,x,\boldsymbol{z}) \approx t(x,x,\boldsymbol{z})\}$$

We call (s,t) a pair of consensus oriented terms.

Higher commutator generation on Mal'cev algebras

Consensus oriented terms

The pair (s, t) or terms is called consensus oriented for V if for all $i \in \underline{n}$, we have we have

$$V \models x_i \approx y_i \Rightarrow s(x_1, y_1, \ldots, x_n, y_n, \boldsymbol{z}) \approx t(x_1, y_1, \ldots, x_n, y_n, \boldsymbol{z}).$$

Examples:

- for rings, $s = \prod_{i=1}^{n}(x_i - y_i)$ and $t = 0$ are consensus oriented.
- $s = (x_1 - y_1) \star (x_2 - y_2) \ - \ 0 \star (x_2 - y_2) \ + \ 0 \star 0 \ - \ (x_1 - y_1) \star 0$ and $t = 0$ are consensus oriented for groups with a binary multiplication $\star$.

Higher Commutator Generation

Theorem [EA and Ž. Semanišinová, 2022]. Let V be a variety with Mal'cev term. Then

$$[\Theta_{\mathbf{A}}(a_1, b_1), \ldots, \Theta_{\mathbf{A}}(a_n, b_n)] = \{\big(s^{\mathbf{A}}(a_1, b_1, \ldots, a_n, b_n, \boldsymbol{e}), t^{\mathbf{A}}(a_1, b_1, \ldots, a_n, b_n, \boldsymbol{e})\big) \mid$$
$$m \in \mathbb{N},\ s, t \text{ are } (n+m)\text{-ary terms},\ (s,t) \text{ is consensus oriented for } V, \boldsymbol{e} \in A^m\}.$$

Higher Commutator Generation

Theorem. Let V be a variety with Mal'cev term. Then

$$[\Theta_{\mathbf{A}}(a_1, b_1), \ldots, \Theta_{\mathbf{A}}(a_n, b_n)] = \{\big(s^{\mathbf{A}}(a_1, b_1, \ldots, a_n, b_n, \boldsymbol{e}), t^{\mathbf{A}}(a_1, b_1, \ldots, a_n, b_n, \boldsymbol{e})\big) \mid m \in \mathbb{N},\ s, t \text{ are } (n+m)\text{-ary terms},\ (s,t) \text{ is consensus oriented for } V, \boldsymbol{e} \in A^m\}.$$

Corollary.

$$[\Theta_{\mathbf{A}}(a_1, b_1), \Theta_{\mathbf{A}}(a_2, b_2)] = \{\big(s^{\mathbf{A}}(a_1, b_1, a_2, b_2, \boldsymbol{e}), t^{\mathbf{A}}(a_1, b_1, a_2, b_2, \boldsymbol{e})\big) \mid V \models s(x_1, x_1, y_1, y_2, \boldsymbol{z}) \approx (x_1, x_1, y_1, y_2, \boldsymbol{z}) \wedge t(x_1, x_2, y_1, y_1, \boldsymbol{z}) \approx s(x_1, x_2, y_1, y_1, \boldsymbol{z})\}.$$

Higher Commutator Generation

Theorem. Let V be a variety with Mal'cev term. Then

$$[\Theta_{\mathbf{A}}(a_1, b_1), \dots, \Theta_{\mathbf{A}}(a_n, b_n)] = \{(s^{\mathbf{A}}(a_1, b_1, \dots, a_n, b_n, \boldsymbol{e}), t^{\mathbf{A}}(a_1, b_1, \dots, a_n, b_n, \boldsymbol{e})) \mid m \in \mathbb{N},\ s, t \text{ are } (n+m)\text{-ary terms},\ (s,t) \text{ is consensus oriented for } V, \boldsymbol{e} \in A^m\}.$$

- V can be any variety in which we have a Mal'cev term.
- If $(G, +, -, 0)$ is a group,

$$V := \mathrm{Mod}(0 + x \approx x,\ x + (-x) \approx 0,\ (x + (-y)) + y \approx 0)$$

is a choice that forces $(x + (-y)) + z$ to be a Mal'cev term, but does not imply $x + 0 \approx x$.

T. Hillenbrand and B. Löchner, Waldmeister, 1999.

Consensus oriented terms are universal

Consensus oriented terms are universal

Theorem. Let V be a variety with a Mal'cev term d. Every subvariety W of V can be defined using

$$d(x, x, y) \approx y \approx d(y, x, x),$$

and, in addition, only identities of the form

- $u(x_1) \approx v(x_1)$ (unary)
- $u(x_1, y_1, \ldots, x_n, y_n) \approx v(x_1, y_1, \ldots, x_n, y_n)$, where (u, v) is consensus oriented in V.

- Lemma 14.5 from [Freese, McKenzie, Commutator Theory for Congruence Modular Varieties, 1987] does the same using commutator terms instead, but requires V to be nilpotent.
- Origins: H. Neumann, G. Higman: expressing group identities using products of commutators.

An application of commutator generation

$[\Theta_{\mathbf{A}}(a,b),1]$ can be parametrized:

Theorem. $\mathbf{A}$ finite algebra with Mal'cev term[1]. Then there exist $m \in \mathbb{N}$ and $t \in \mathrm{Clo}_{m+1}(\mathbf{A})$ such that

- $\mathbf{A} \models \exists \boldsymbol{z}\, \forall x, y : t(x, \boldsymbol{z}) \approx t(y, \boldsymbol{z})$,
- for all $a, b \in A^k$: $[\Theta_{\mathbf{A}}(a,b),1] = \{(t(a,\boldsymbol{w}), t(b,\boldsymbol{w})) \mid \boldsymbol{w} \in A^m\}$.

Main ingredient in proving

Theorem. For a nonabelian finite Mal'cev algebra $\mathbf{A}$ of finite type, checking the validity of quasi-identities in $\mathbf{A}$ is coNP-complete.

E. Aichinger and S. Grünbacher. The Complexity of Checking Quasi-Identities over Finite Algebras with a Mal'cev Term, STACS 2023

[1] Assumptions added after presentation.