Strong subalgebras
and
how to use them

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Prague

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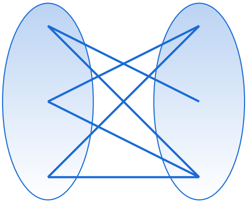

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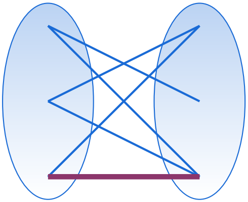

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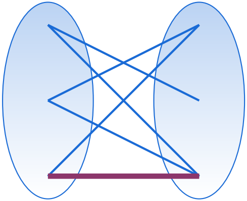

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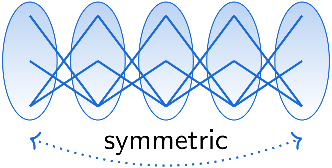

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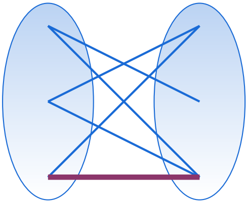

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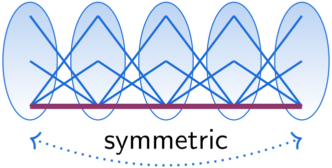

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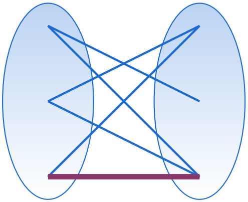

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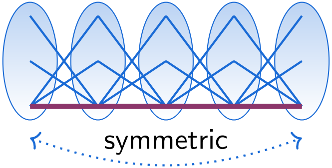

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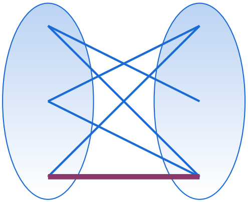

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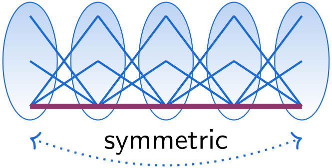

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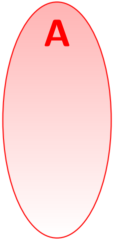

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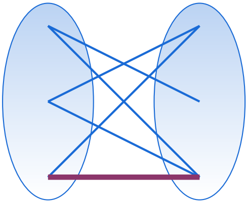

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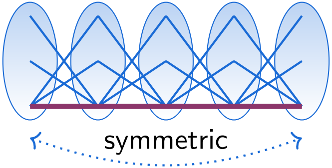

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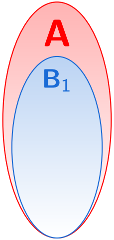

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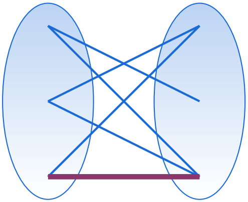

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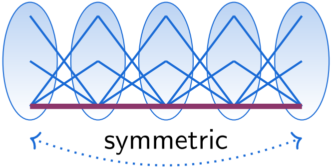

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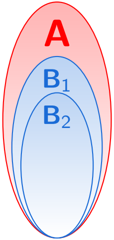

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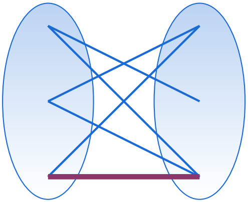

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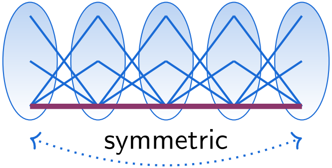

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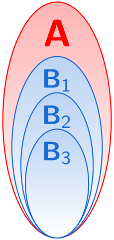

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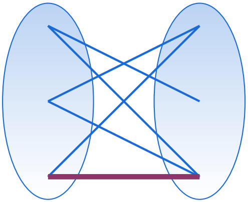

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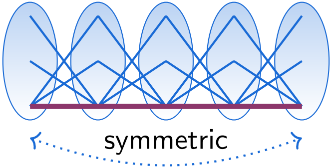

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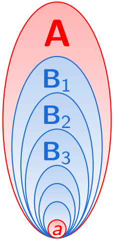

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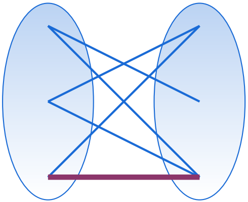

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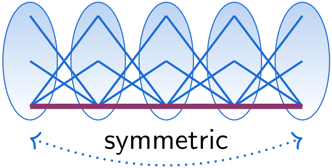

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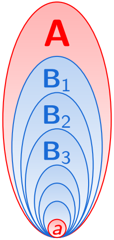

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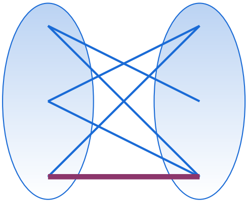

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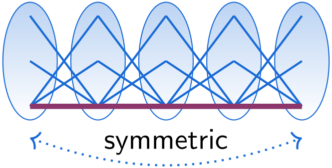

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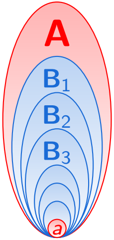

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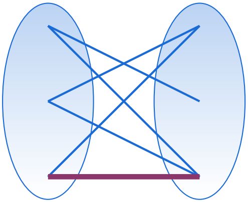

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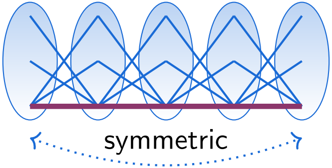

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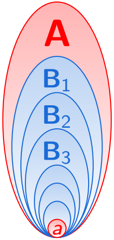
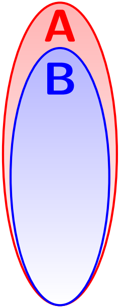
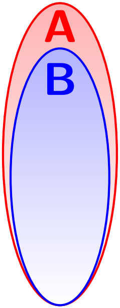




B absorbs **A** (write $\mathbf{B} \trianglelefteq \mathbf{A}$) if $\mathbf{B} \leq \mathbf{A}$ and $\exists t \in \mathrm{Clo}(\mathbf{A})$ s.t.

$$\forall i\colon\ t(\underbrace{B, \ldots, B, A}_{i}, B, \ldots, B) \subseteq B.$$



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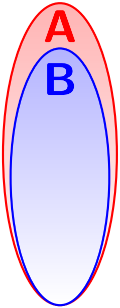
Write $\mathbf{B} \trianglelefteq_k \mathbf{A}$ if t is of arity k.

Absorbing Subalgebras



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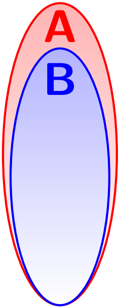

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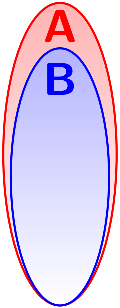

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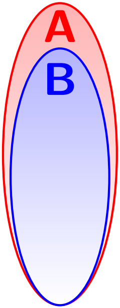

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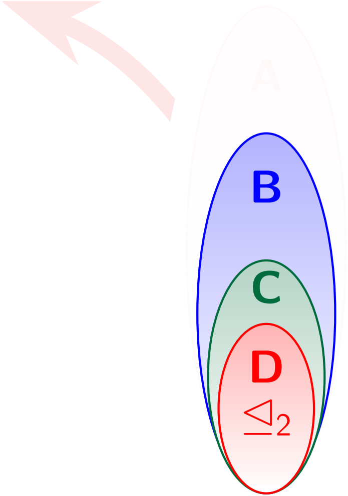
$\mathbf{C} <_{\mathcal{S}} \mathbf{B} \Leftrightarrow \mathbf{C} \leq \mathbf{B} \wedge \exists \mathbf{D} \lhd_2 \mathbf{B}$ s.t. $D \subseteq C$.



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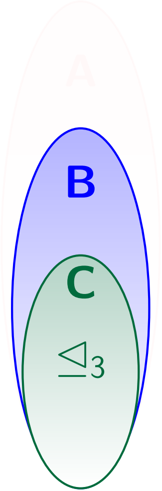

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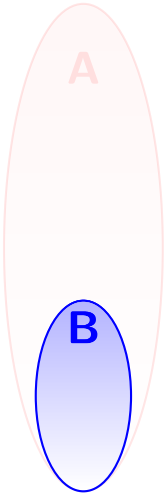

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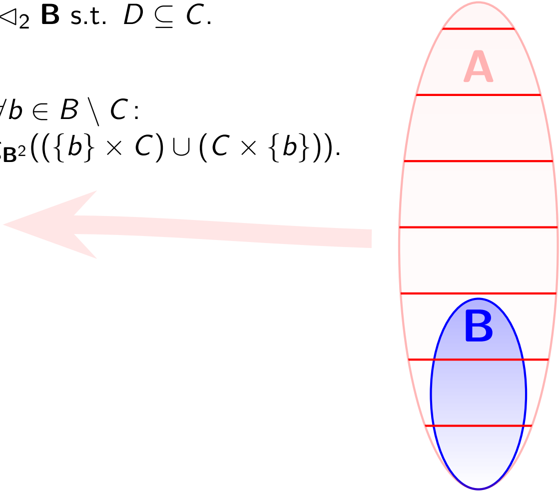

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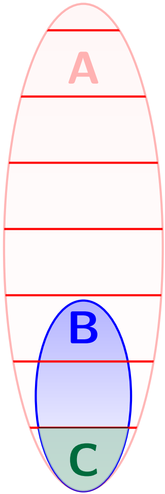

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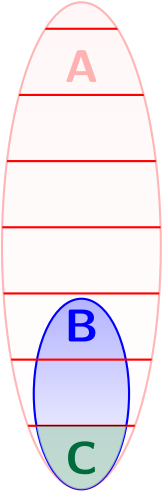

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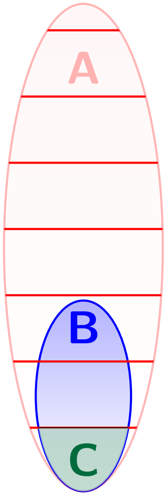

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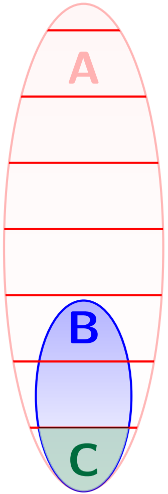

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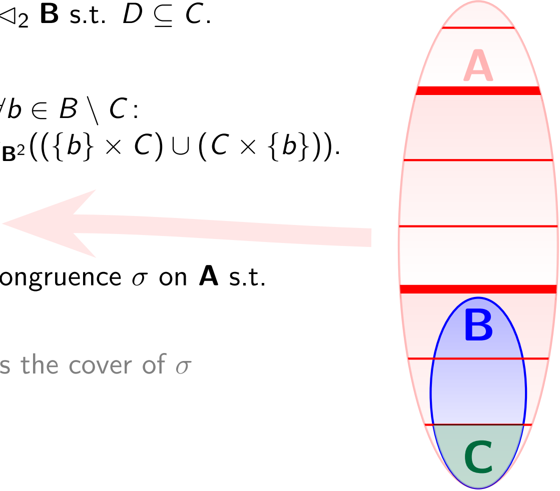

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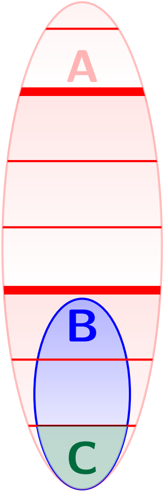

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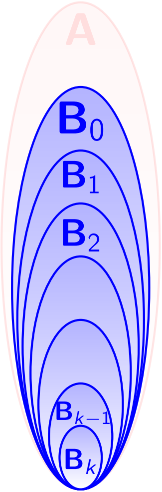

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where $T_i \in \{\mathcal{S}, \mathcal{C}, \mathcal{D}\}$.

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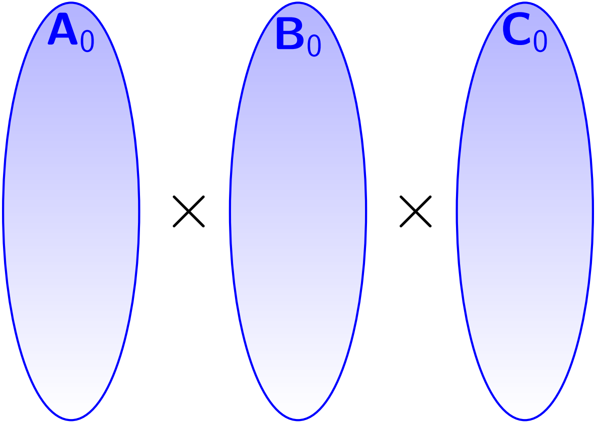

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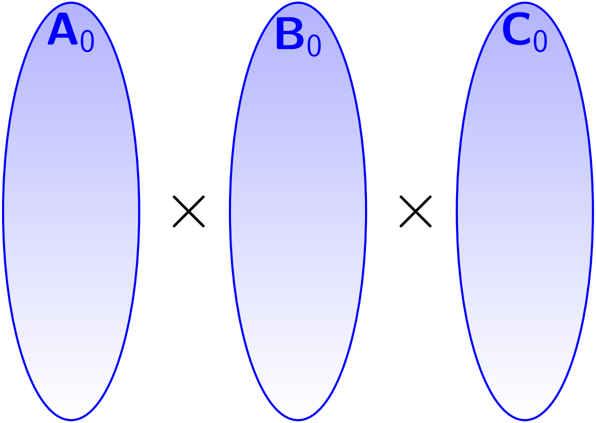

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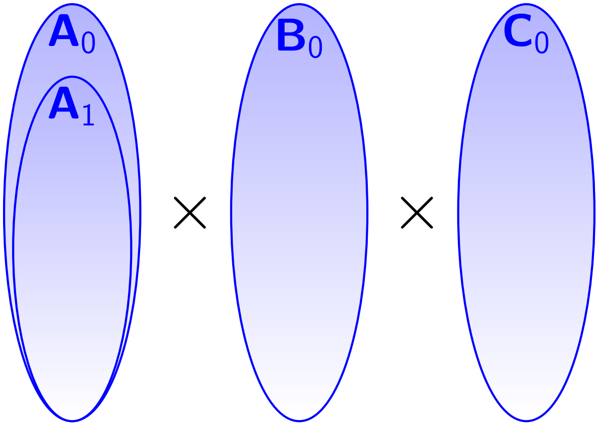

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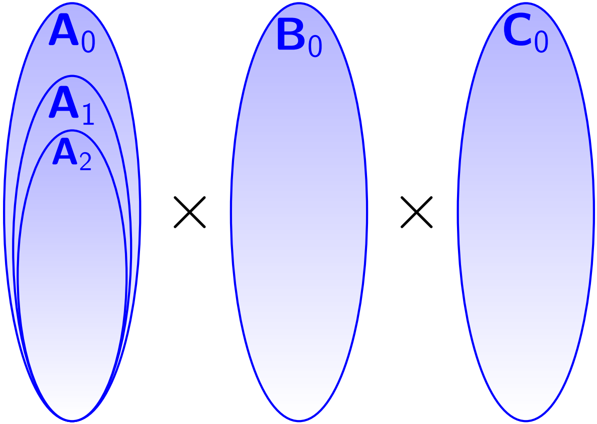

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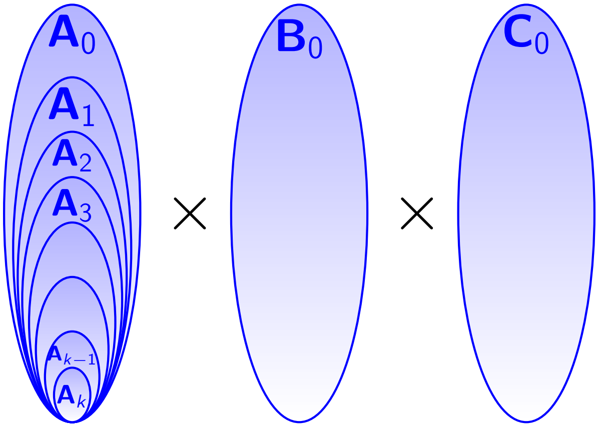

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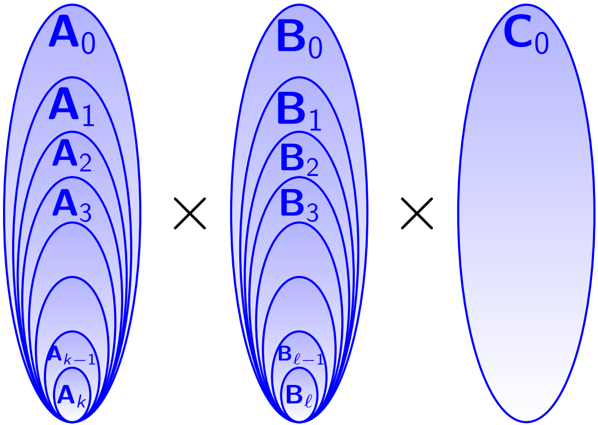

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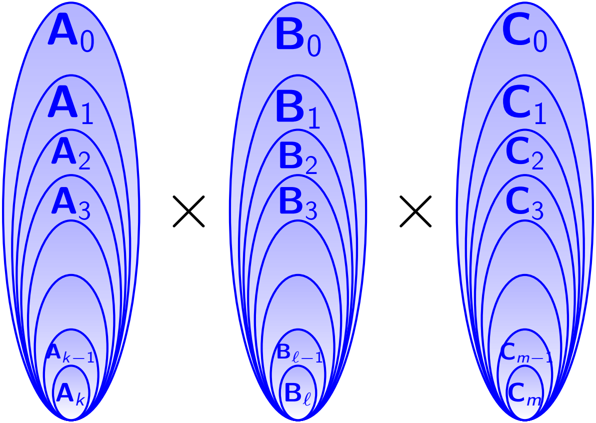

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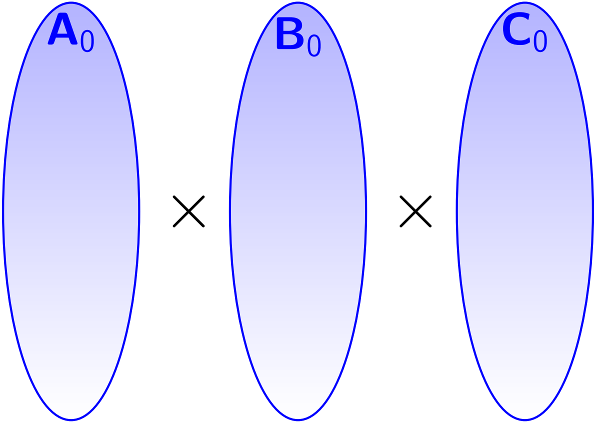

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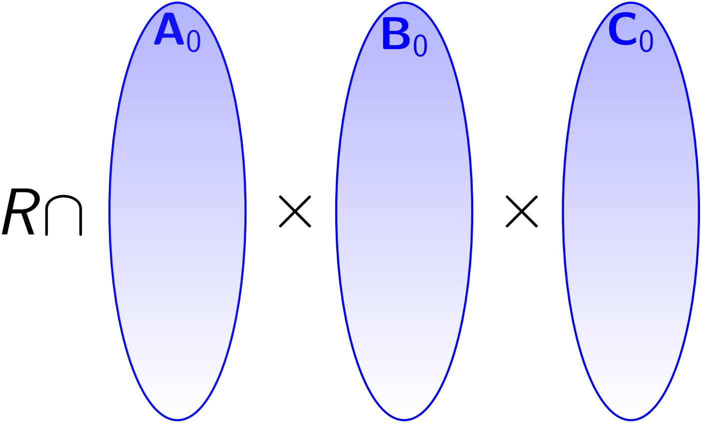

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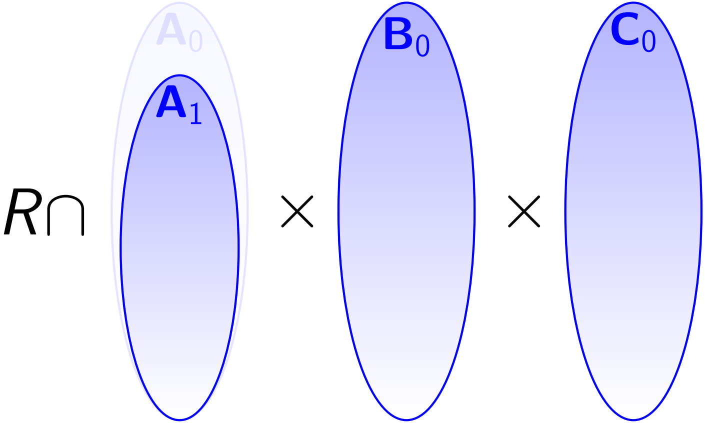

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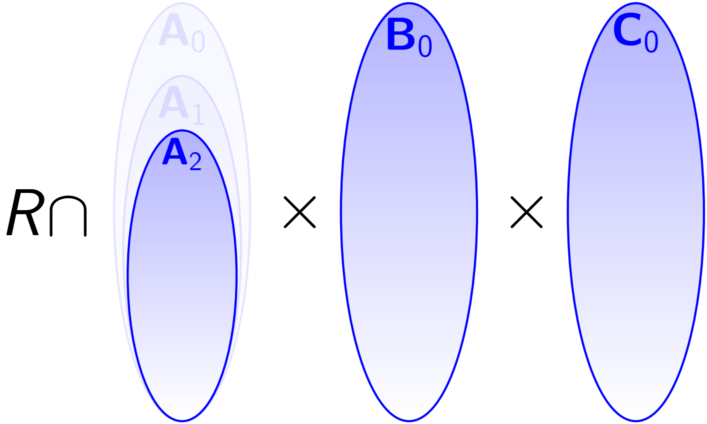

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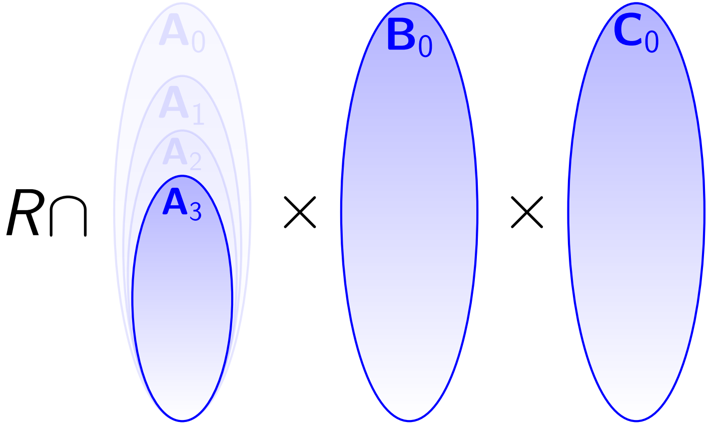

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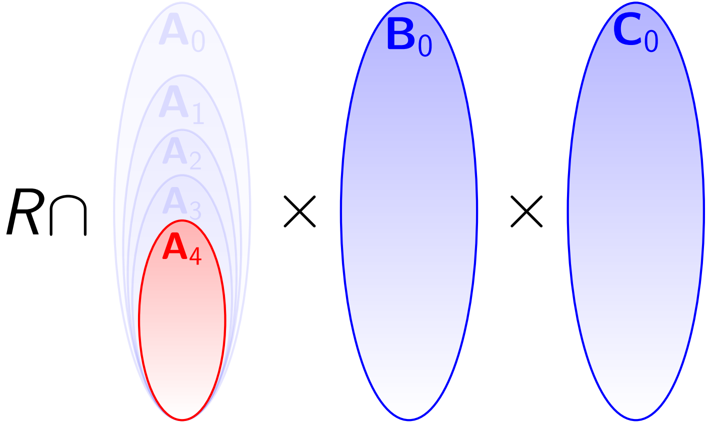

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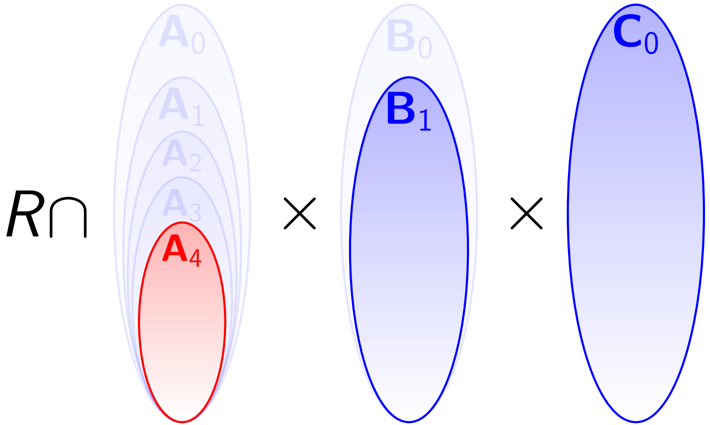

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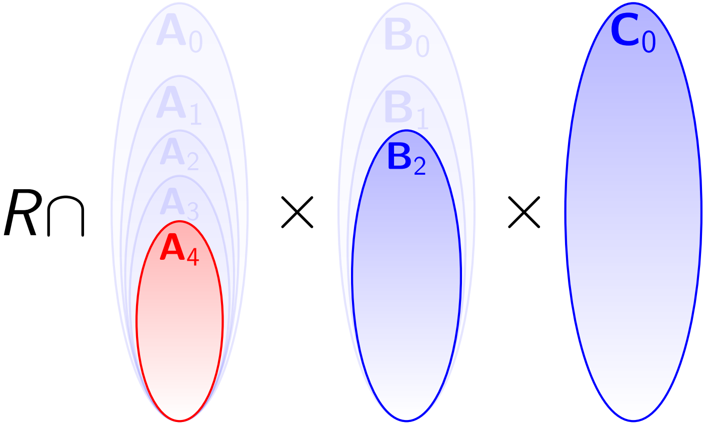

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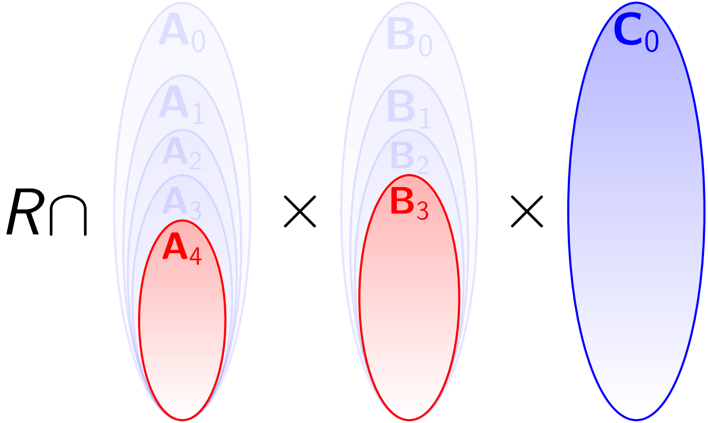

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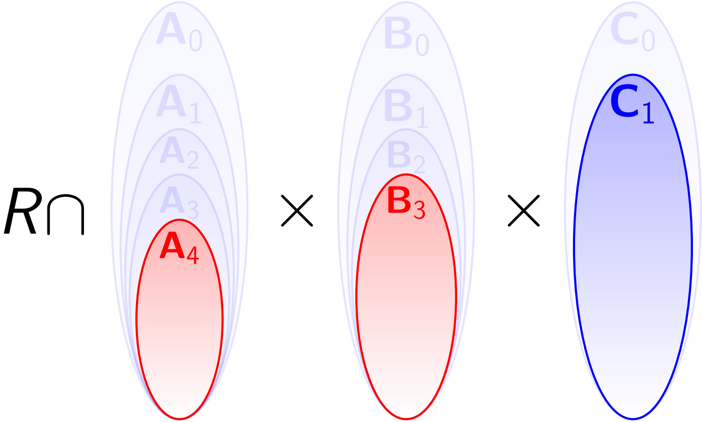

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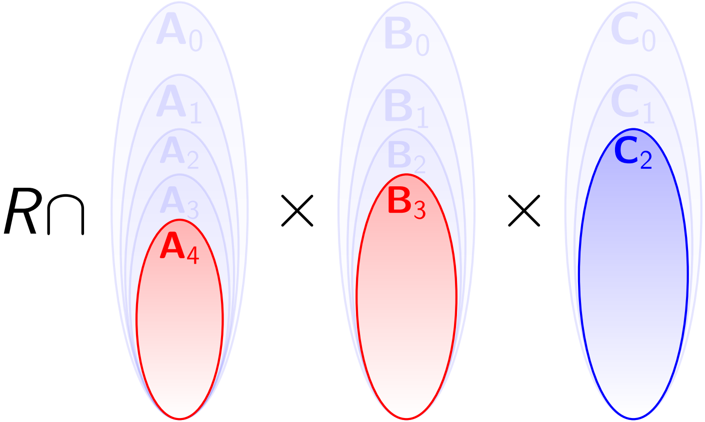

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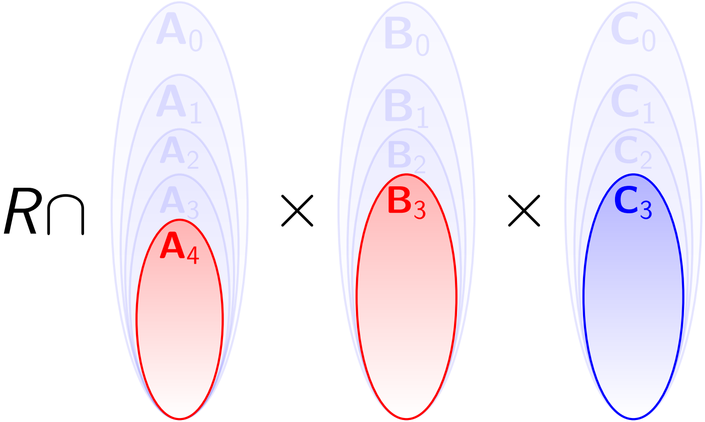

Helly property

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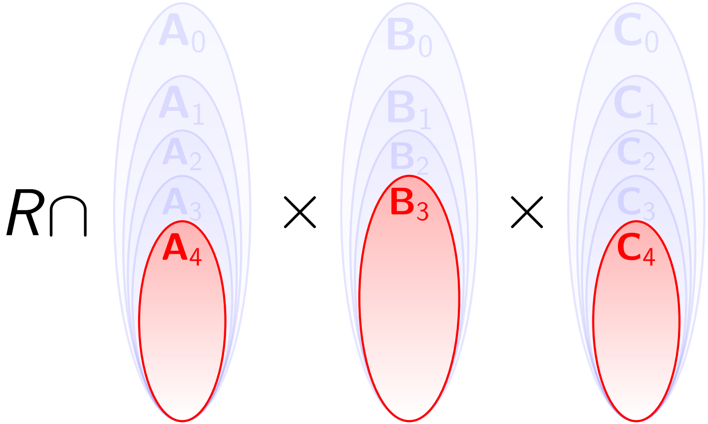

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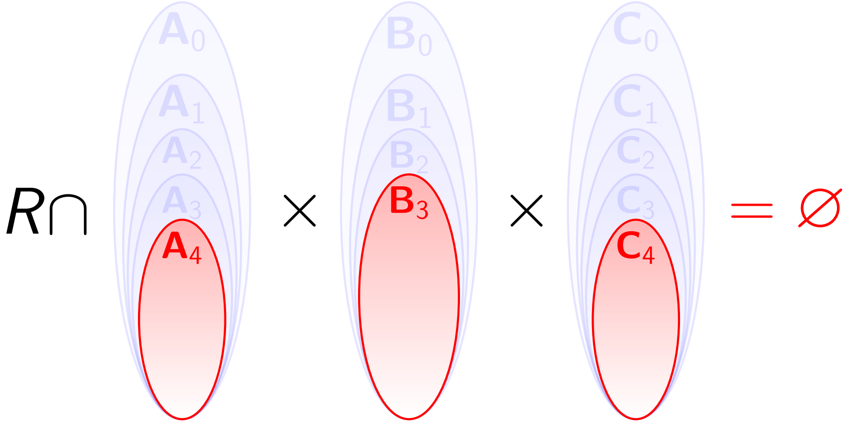

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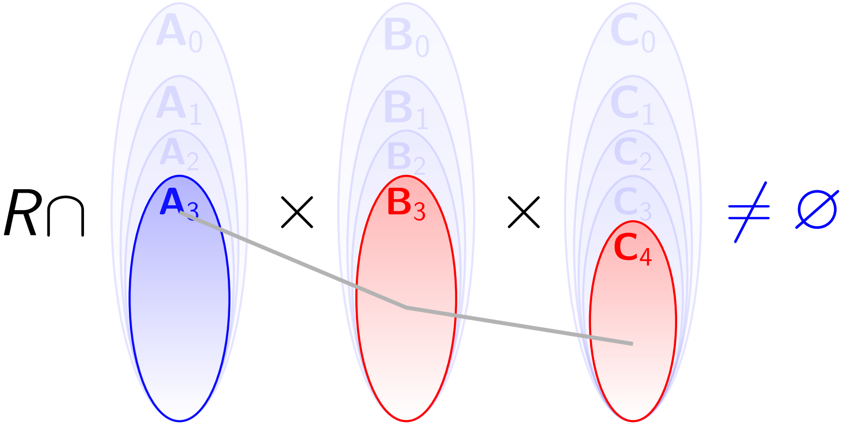

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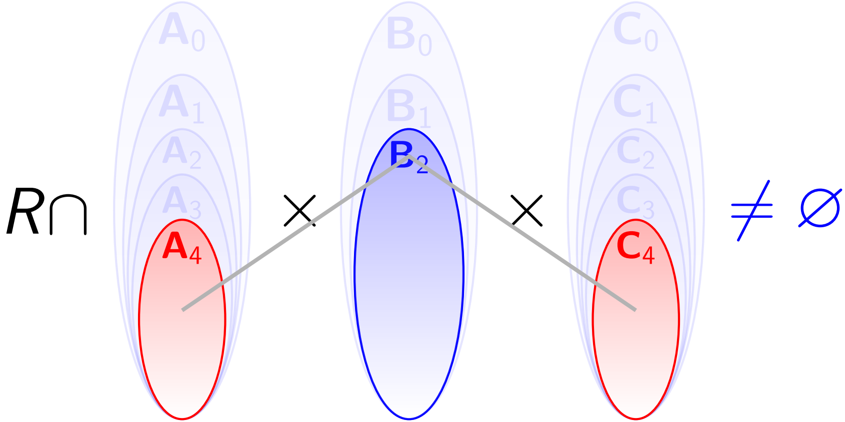

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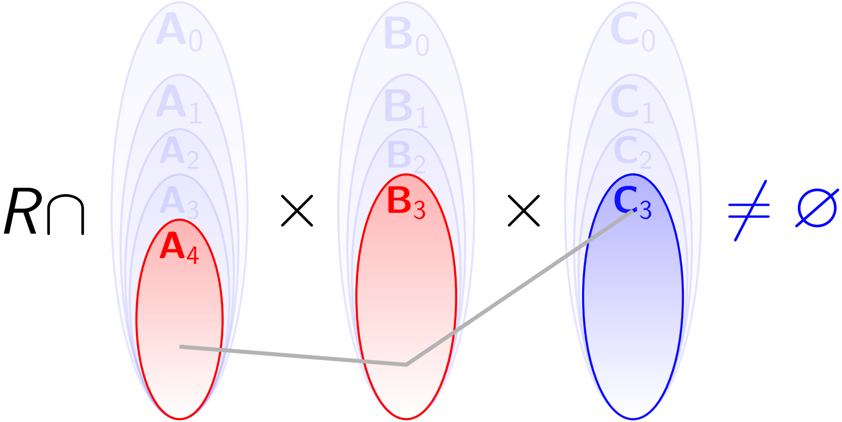

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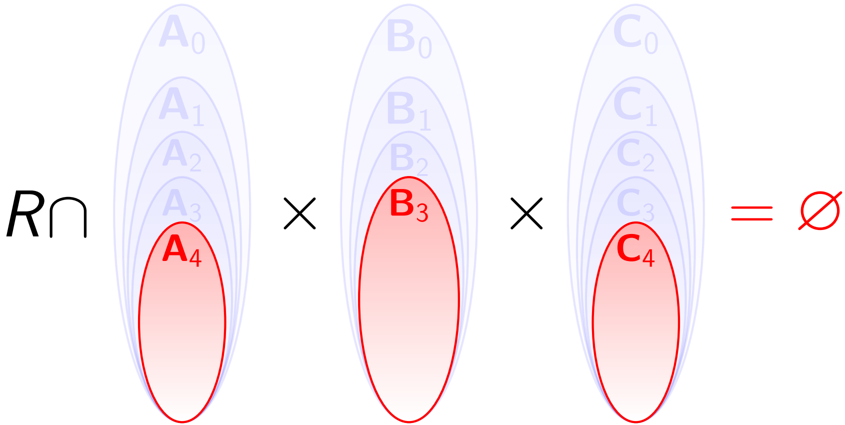
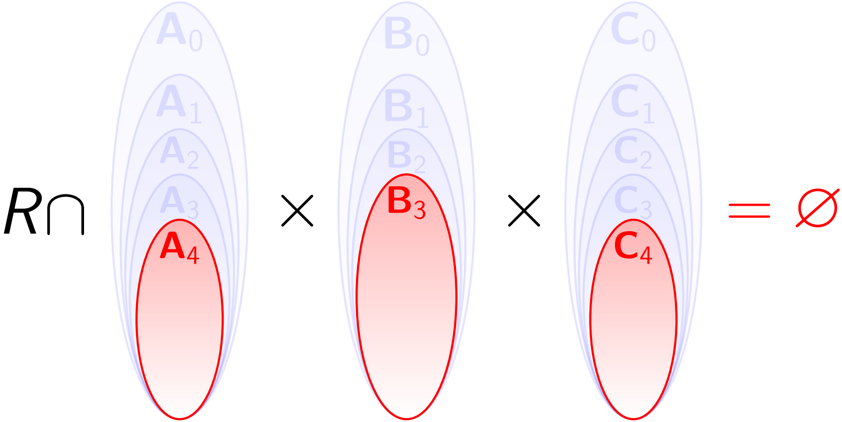


Helly property

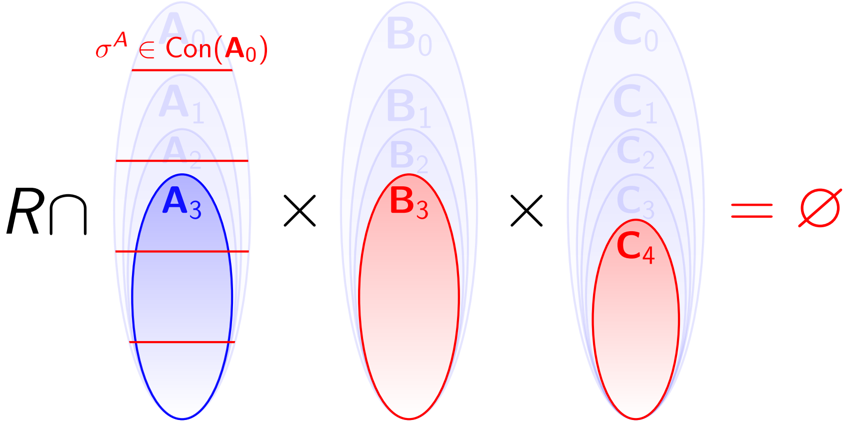
$$\mathbf{R} \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



Then $\mathbf{A}_4 <^{A_0}_{\mathcal{D}} \mathbf{A}_3$

Helly property

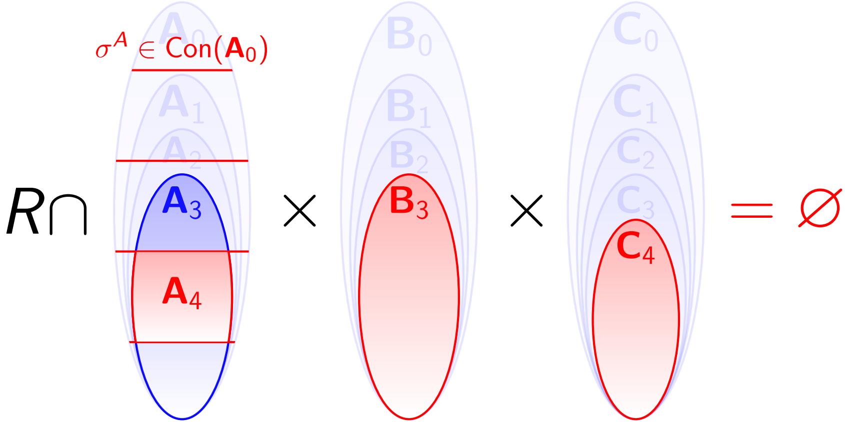
$$\mathbf{R} \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



Then $\quad \mathbf{A}_4 <_{\mathcal{D}}^{A_0} \mathbf{A}_3$

Helly property

$$\mathbf{R} \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



Then $\mathbf{A}_4 <_{\mathcal{D}}^{A_0} \mathbf{A}_3$

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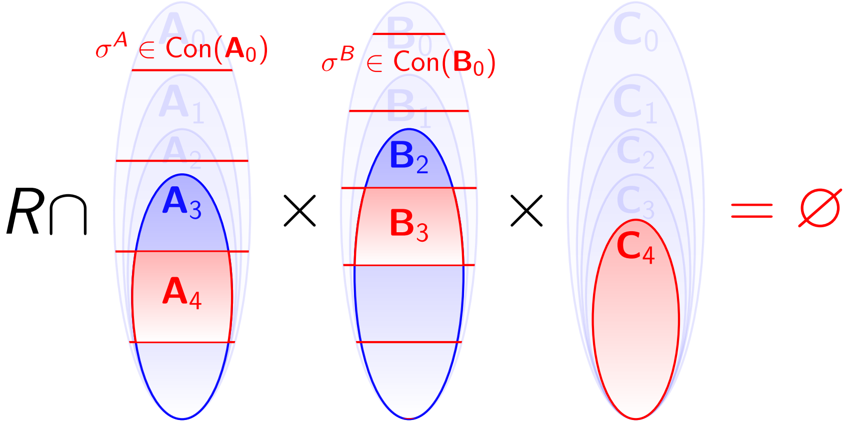
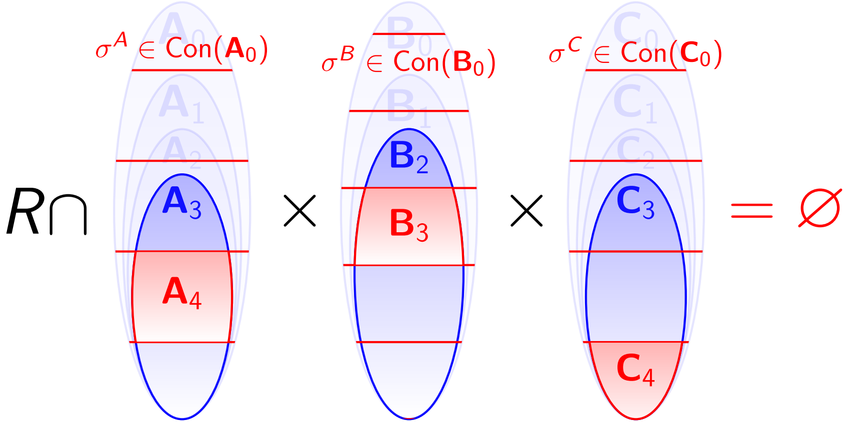
$$\mathbf{R} \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



Then $\quad \mathbf{A}_4 <_{\mathcal{D}}^{A_0} \mathbf{A}_3 \qquad \mathbf{B}_3 <_{\mathcal{D}}^{B_0} \mathbf{B}_2$

Helly property

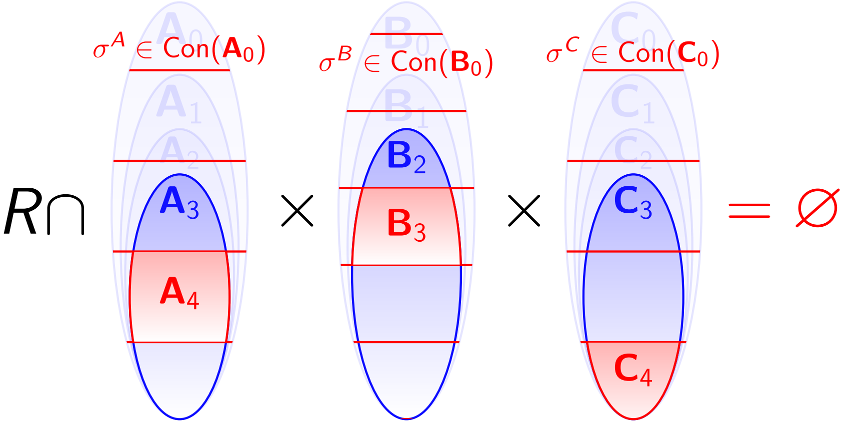
$\mathbf{R} \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0$, $\mathbf{A}_k \lll^{A_0} \mathbf{A}_0$, $\mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0$, $\mathbf{C}_m \lll^{C_0} \mathbf{C}_0$.



Then $\mathbf{A}_4 <^{A_0}_{\mathcal{D}} \mathbf{A}_3$ $\quad$ $\mathbf{B}_3 <^{B_0}_{\mathcal{D}} \mathbf{B}_2$ $\quad$ $\mathbf{C}_4 <^{C_0}_{\mathcal{D}} \mathbf{C}_3$

Helly property

$$\mathbf{R} \le_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



Then $\quad \mathbf{A}_4 <_{\mathcal{D}}^{A_0} \mathbf{A}_3 \quad \mathbf{B}_3 <_{\mathcal{D}}^{B_0} \mathbf{B}_2 \quad \mathbf{C}_4 <_{\mathcal{D}}^{C_0} \mathbf{C}_3$

There exist bridges between congruences σ^A, σ^B, and σ^C

Properties of $\lll^A$

(Transitivity) $\mathbf{D} \lll^A \mathbf{C} \lll^A \mathbf{B} \Rightarrow \mathbf{D} \lll^A \mathbf{B}$

(Intersection) $\mathbf{C}, \mathbf{D} \lll^A \mathbf{B} \wedge \mathbf{C} \cap \mathbf{D} \neq 0 \Rightarrow \mathbf{C} \cap \mathbf{D} \lll^A \mathbf{B}$

(Propagation) if $f \colon \mathbf{A} \to \mathbf{A}'$ is a surjective homomorphism,

(Pushforward) $\mathbf{C} \lll^A \mathbf{B} \Rightarrow f(\mathbf{C}) \lll^{A'} \mathbf{f}(B)$

(Pullback) $\mathbf{C}' \lll^{A'} \mathbf{B}' \Rightarrow f^{-1}(\mathbf{C}') \lll^A \mathbf{f}^{-1}(B')$

(Ubiquity) if $\mathbf{B} \lll^A \mathbf{A}$ and $|\mathbf{B}| > 1$ then $\exists \mathbf{C} \lll^A \mathbf{B}$.

(Helly)

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(Helly) Suppose

- $R \leq_{sd} \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$, $n \geq 2$,

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then

1. either $T_i = \mathcal{C}$ for every i and $n = 2$,

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then

1. either $T_i = \mathcal{C}$ for every i and $n = 2$,
2. or $T_i = \mathcal{D}$ for every i

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then

1. either $T_i = \mathcal{C}$ for every i and $n = 2$,
2. or $T_i = \mathcal{D}$ for every i + for all i, j there exists a bridge δ from σ_i to σ_j such that $\mathrm{tr}(\delta) = \mathrm{pr}_{i,j}(R)$

Bridges

Bridges

$\sigma_1 \in \mathsf{Con}(\mathbf{A}_1)$, $\sigma_2 \in \mathsf{Con}(\mathbf{A}_2)$.

Bridges

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TFAE

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TFAE

- there exists a bridge from σ_1 to σ_2;

Bridges

$\sigma_1 \in \mathrm{Con}(\mathbf{A}_1)$, $\sigma_2 \in \mathrm{Con}(\mathbf{A}_2)$.

TFAE

- there exists a bridge from σ_1 to σ_2;
- there exists a bridge from $0_{\mathbf{A}_1/\sigma_1}$ to $0_{\mathbf{A}_2/\sigma_2}$;

Bridges

$\sigma_1 \in \mathrm{Con}(\mathbf{A}_1)$, $\sigma_2 \in \mathrm{Con}(\mathbf{A}_2)$.

TFAE

- there exists a bridge from σ_1 to σ_2;
- there exists a bridge from $0_{\mathbf{A}_1/\sigma_1}$ to $0_{\mathbf{A}_2/\sigma_2}$;
- $\mathbf{A}_1/\sigma_1$ and $\mathbf{A}_2/\sigma_2$ are similar.

Bridges

$\sigma_1 \in \mathrm{Con}(\mathbf{A}_1)$, $\sigma_2 \in \mathrm{Con}(\mathbf{A}_2)$.
$\delta \leq \mathbf{A}_1^2 \times \mathbf{A}_2^2$ is a bridge from σ_1 to σ_2 if

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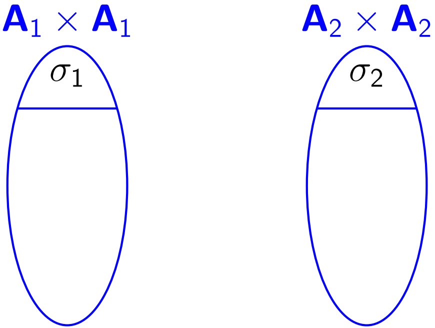

TFAE

- there exists a bridge from σ_1 to σ_2;
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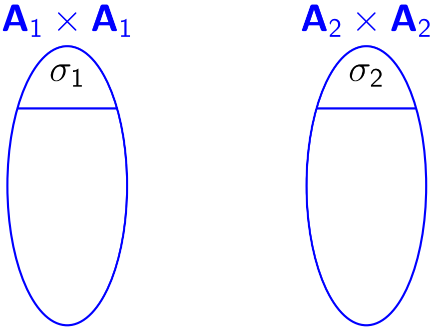
Bridges

$\sigma_1 \in \mathrm{Con}(\mathbf{A}_1)$, $\sigma_2 \in \mathrm{Con}(\mathbf{A}_2)$.
$\delta \leq \mathbf{A}_1^2 \times \mathbf{A}_2^2$ is a bridge from σ_1 to σ_2 if

1. $(a_1, a_2, a_3, a_4) \in \delta$



TFAE

- there exists a bridge from σ_1 to σ_2;
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Bridges

$\sigma_1 \in \mathrm{Con}(\mathbf{A}_1)$, $\sigma_2 \in \mathrm{Con}(\mathbf{A}_2)$.
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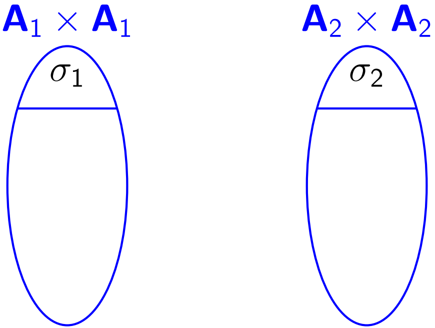
1. $$\begin{array}{cccc} (a_1, & a_2, & a_3, & a_4) \in \delta \\ \sigma_1 \wr & \sigma_1 \wr & \wr \sigma_2 & \wr \sigma_2 \\ (b_1, & b_2, & b_3, & b_4) & \end{array}$$



TFAE

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Bridges

$\sigma_1 \in \mathrm{Con}(\mathbf{A}_1)$, $\sigma_2 \in \mathrm{Con}(\mathbf{A}_2)$.
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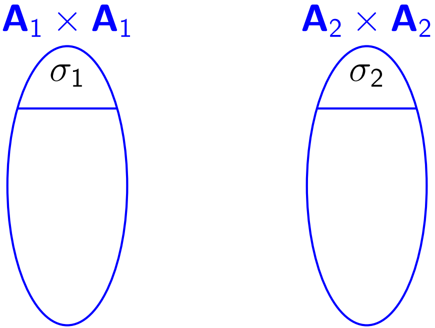
1. $$\begin{array}{c}(a_1, a_2, a_3, a_4) \in \delta \\ \sigma_1 \quad \sigma_1 \qquad \sigma_2 \quad \sigma_2 \qquad \Downarrow \\ (b_1, b_2, b_3, b_4) \in \delta\end{array}$$



TFAE

- there exists a bridge from σ_1 to σ_2;
- there exists a bridge from $0_{\mathbf{A}_1/\sigma_1}$ to $0_{\mathbf{A}_2/\sigma_2}$;
- $\mathbf{A}_1/\sigma_1$ and $\mathbf{A}_2/\sigma_2$ are similar.

Bridges

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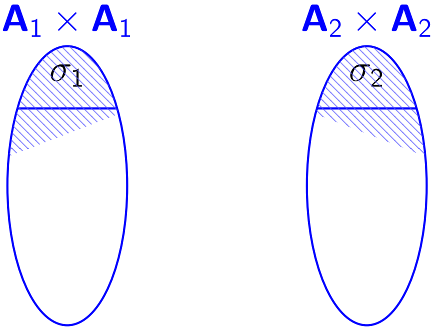
2. $\mathrm{pr}_{1,2}(\delta) \supsetneq \sigma_1$,
   $\mathrm{pr}_{3,4}(\delta) \supsetneq \sigma_2$



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1. $$\begin{array}{c} (a_1, a_2, a_3, a_4) \in \delta \\ {\scriptstyle \sigma_1} \wr\ {\scriptstyle \sigma_1} \wr \quad \wr\ {\scriptstyle \sigma_2} \wr\ {\scriptstyle \sigma_2} \qquad \Downarrow \\ (b_1, b_2, b_3, b_4) \in \delta \end{array}$$
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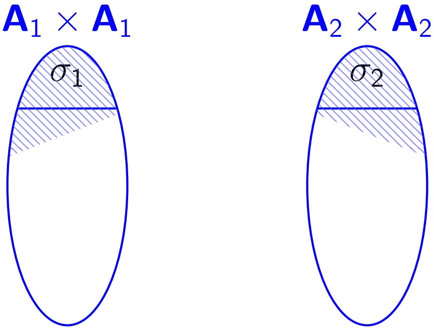

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(a_1, a_2, a_3, a_4) \in \delta \\
{\scriptstyle \sigma_1 \wr\; \sigma_1 \wr \quad \wr\; \sigma_2 \wr\; \sigma_2} \quad \Downarrow \\
(b_1, b_2, b_3, b_4) \in \delta
\end{array}$$
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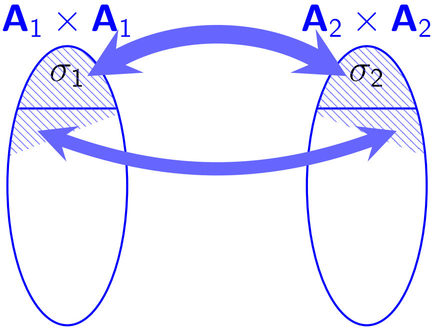
$$(a_1, a_2) \in \sigma_1 \Leftrightarrow (a_3, a_4) \in \sigma_2.$$



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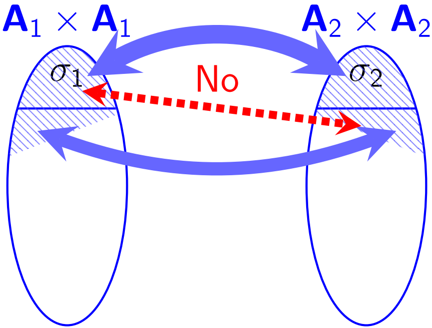

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1. $$\begin{array}{cccccccccc} (& a_1, & a_2, & a_3, & a_4 &) & \in \delta & \\ & \sigma_1 \wr & \sigma_1 \wr & \wr \sigma_2 & \wr \sigma_2 & & & \Downarrow \\ (& b_1, & b_2, & b_3, & b_4 &) & \in \delta & \end{array}$$
2. $\mathrm{pr}_{1,2}(\delta) \supsetneq \sigma_1$,
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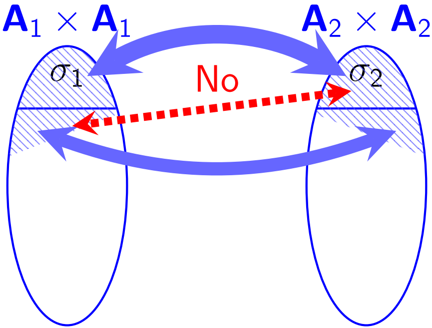

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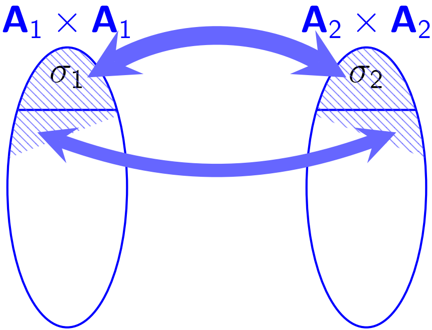

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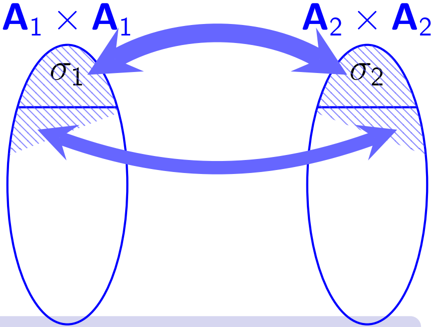

Example

Bridges

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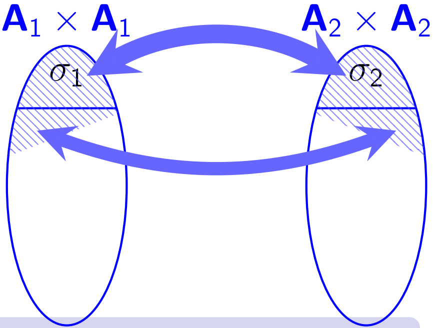

Example

$\mathbf{A}_1 = \mathbf{A}_2 = (\mathbb{Z}_4, x - y + z)$

Bridges

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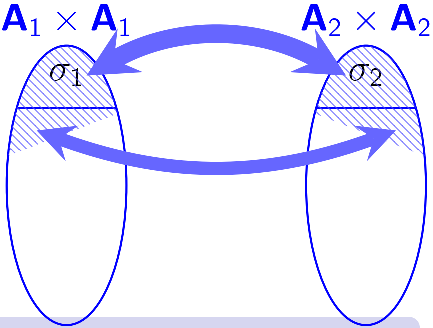

Example

$\mathbf{A}_1 = \mathbf{A}_2 = (\mathbb{Z}_4, x - y + z)$
$\delta := \{(x_1, x_2, x_3, x_4) \in \mathbb{Z}_4^4 \mid x_1 - x_2 = 2x_3 - 2x_4\}$,

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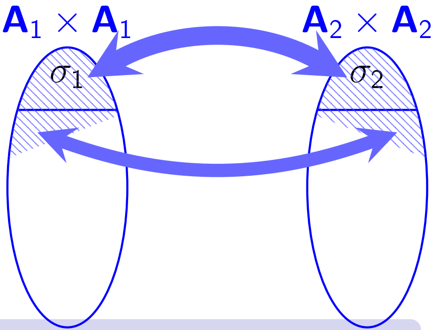

Example

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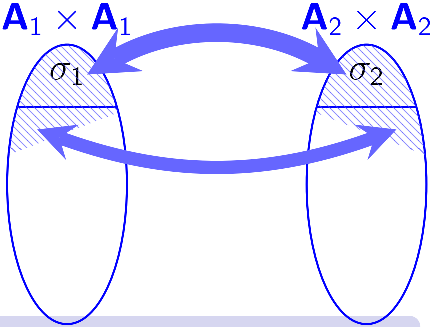

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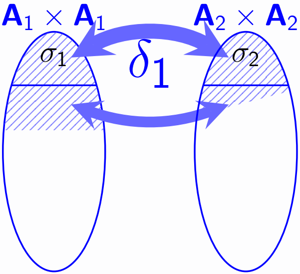
- Algebras $\mathbb{Z}_4$ and $\mathbb{Z}_2$ are similar.

Composition of bridges

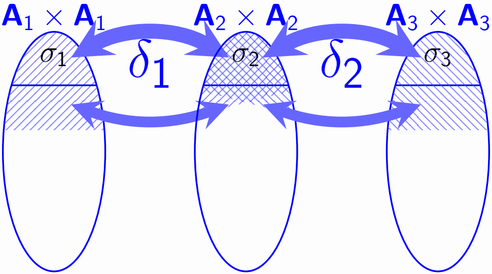
Composition of bridges

- δ_1 is a bridge from σ_1 to σ_2



Composition of bridges

- δ_1 is a bridge from σ_1 to σ_2
- δ_2 is a bridge from σ_2 to σ_3



Composition of bridges

- δ_1 is a bridge from σ_1 to σ_2
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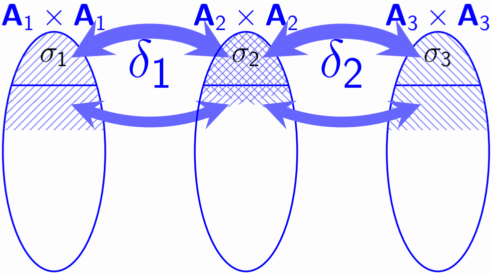
- $\delta := \delta_1 \circ \delta_2$ is a bridge from σ_1 to σ_3 (if σ_2 is irreducible)

  $\delta(x_1, x_2, z_1, z_2) := \exists y_1 \exists y_2\ \delta_1(x_1, x_2, y_1, y_2) \wedge \delta_2(y_1, y_2, z_1, z_2)$



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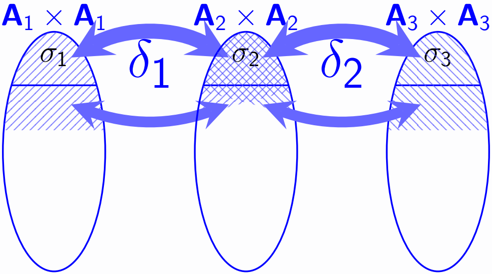

For a bridge δ from σ_1 on $\mathbf{A}_1$ to σ_2 on $\mathbf{A}_2$
the trace of a bridge δ, denoted $\mathrm{tr}(\delta)$, is a subuniverse $\mathbf{A}_1 \times \mathbf{A}_2$ defined by
$\mathrm{tr}(\delta)(x, y) = \delta(x, x, y, y)$.

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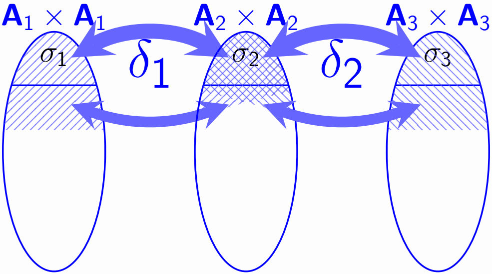

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- $\delta = \delta_1 \circ \delta_2 \Longrightarrow \mathrm{tr}(\delta) = \mathrm{tr}(\delta_1) \circ \mathrm{tr}(\delta_2)$

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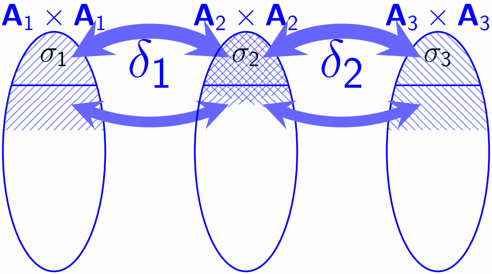

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- $\delta = \delta_1 \circ \delta_2 \implies \mathrm{tr}(\delta) = \mathrm{tr}(\delta_1) \circ \mathrm{tr}(\delta_2)$
- for every irreducible congruence σ there exists a bridge δ from σ to σ with the largest reflexive trace,

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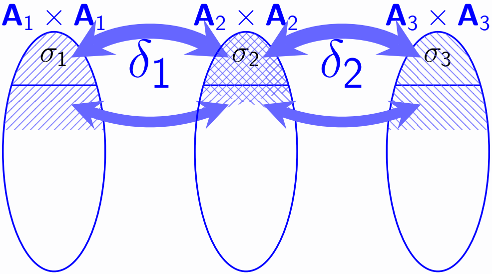

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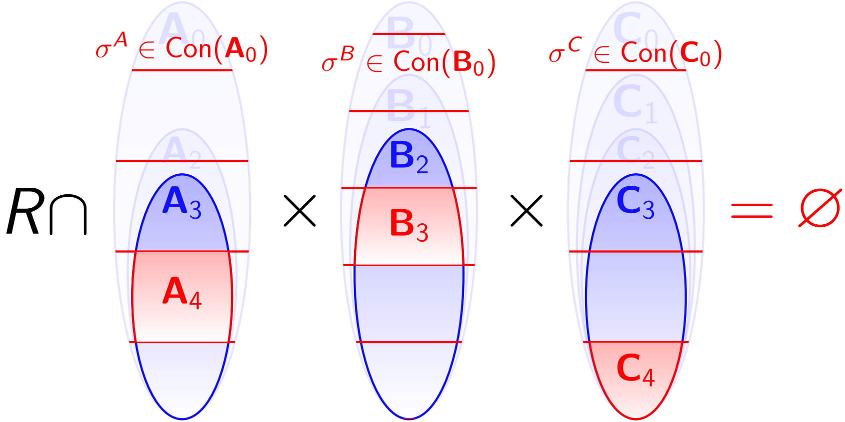
the trace of a bridge δ, denoted $\mathrm{tr}(\delta)$, is a subuniverse $\mathbf{A}_1 \times \mathbf{A}_2$ defined by $\mathrm{tr}(\delta)(x, y) = \delta(x, x, y, y)$. Then

- $\delta = \delta_1 \circ \delta_2 \implies \mathrm{tr}(\delta) = \mathrm{tr}(\delta_1) \circ \mathrm{tr}(\delta_2)$
- for every irreducible congruence σ there exists a bridge δ from σ to σ with the largest reflexive trace, moreover, $\mathrm{tr}(\delta)$ is the centralizer of σ^* modulo σ.

Helly property

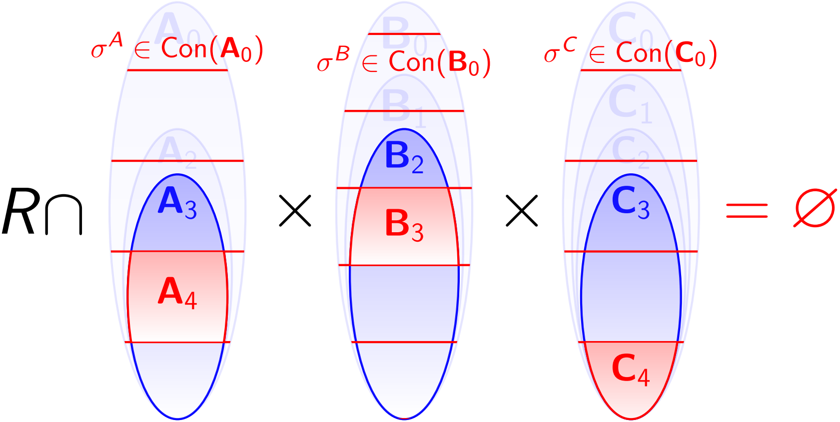
$$\mathbf{R} \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0,\ \mathbf{A}_k \lll^{A_0} \mathbf{A}_0,\ \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0,\ \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



Then $\quad \mathbf{A}_4 <_{\mathcal{D}}^{A_0} \mathbf{A}_3 \qquad \mathbf{B}_3 <_{\mathcal{D}}^{B_0} \mathbf{B}_2 \qquad \mathbf{C}_4 <_{\mathcal{D}}^{C_0} \mathbf{C}_3$

Helly property

$\mathbf{R} \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0$, $\mathbf{A}_k \lll^{A_0} \mathbf{A}_0$, $\mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0$, $\mathbf{C}_m \lll^{C_0} \mathbf{C}_0$.



Then $\mathbf{A}_4 <_{\mathcal{D}}^{A_0} \mathbf{A}_3$ $\quad \mathbf{B}_3 <_{\mathcal{D}}^{B_0} \mathbf{B}_2$ $\quad \mathbf{C}_4 <_{\mathcal{D}}^{C_0} \mathbf{C}_3$

There exists a bridge δ from σ^A to σ^B such that $\mathrm{tr}(\delta) = \mathrm{pr}_{1,2}(R)$

Example: Group S_3

Example: Group $\mathbf{S}_3$

$$S_3 = \{e,\ (12),\ (13),\ (23),\ (123),\ (132)\}.$$

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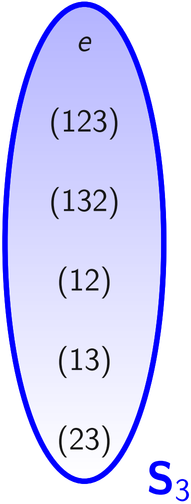

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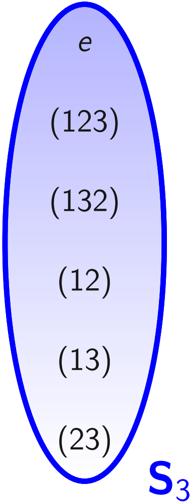
Congruence lattice

$$1_{\mathbf{S}_3}$$
$$|$$
$$\sigma \quad (A_3)$$
$$|$$
$$0_{\mathbf{S}_3}$$



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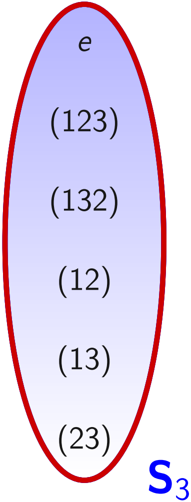

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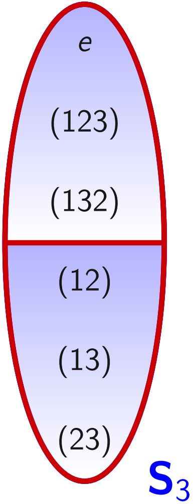
$\mathbf{S}_3/\sigma \cong \mathbb{Z}_2$



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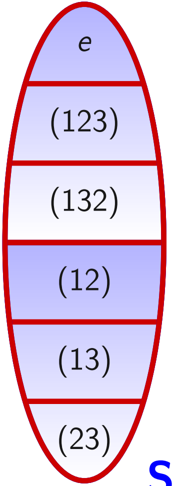

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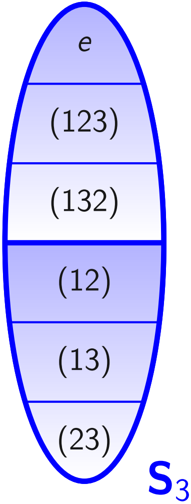

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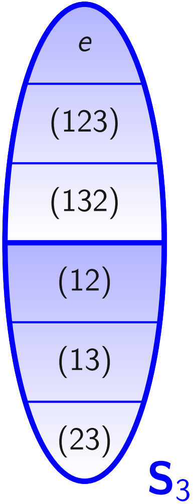
Subuniverses of $\mathbf{S}_3^{id}$ (idempotent reduct):



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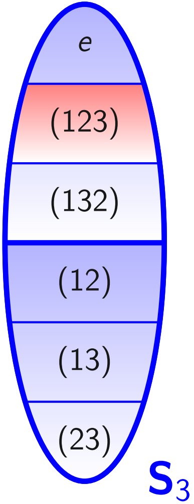

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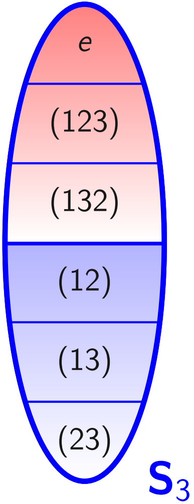

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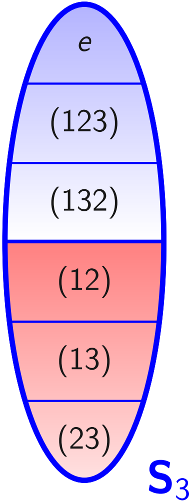

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Congruence lattice

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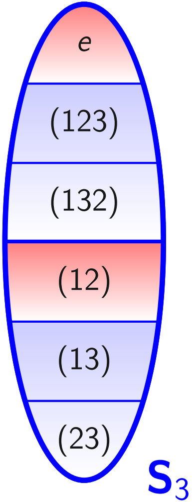

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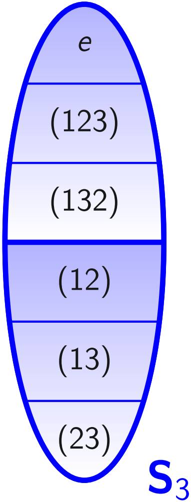

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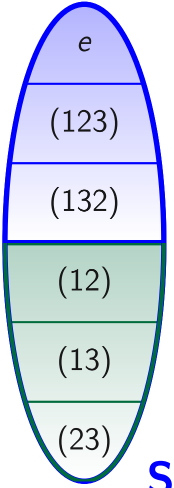

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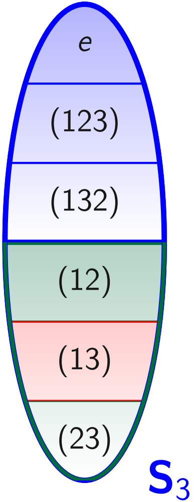

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$|$

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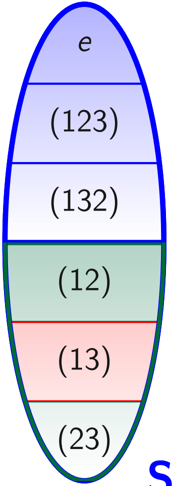

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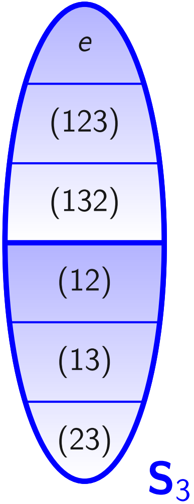

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Bridges:

Example: Group S_3

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Congruence lattice

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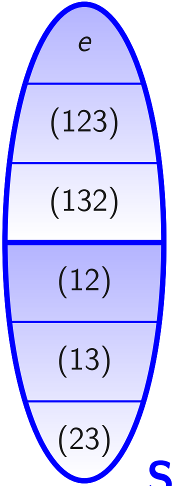

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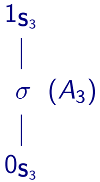

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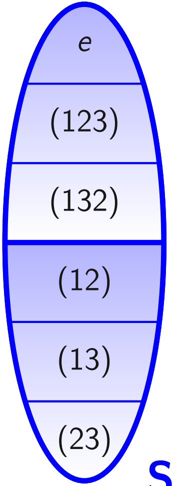

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- $\forall a \in S_3\colon \{a\} \lll \mathbf{S}_3$

Bridges:

- No bridges between $0_{\mathbf{S}_3}$, $0_{\mathbf{Z}_3}$ and $\sigma(\sim 0_{\mathbf{Z}_2})$
- $\mathrm{tr}(\delta) \subseteq \sigma$ for any reflexive bridge δ from $0_{\mathbf{S}_3}$ to $0_{\mathbf{S}_3}$

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$f\colon A^n \to A$ is XY-symmetric if

$$f|_{a,b}\colon \{a, b\}^n \to A \text{ is symmetric for all } a, b \in A$$

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- Does $\mathbf{S}_3^{id}$ have XY-symmetric term operations of arity 5? Yes!

Claim

There exists a term operation f in $\mathbf{S}_3^{id}$ s. t. for all $a, b \in S_3$

- $f|_{a,b}$ is symmetric;
- $f|_{a,b}(b, a, a, a, a) = f|_{a,b}(b, b, b, a, a) = b$ whenever $(a, b) \notin \sigma$;
- $f|_{a,b}(x_1, x_2, x_3, x_4, x_5) = x_1^2 \circ \cdots \circ x_5^2$ whenever $(a, b) \in \sigma$.

Example: Group S_3

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$$
\mathbf{a} = \left(\begin{array}{c}
(13)\\ (13)\\ (13)\\ \vdots\\ (13)\\ \hline
(123)\\ (123)\\ (123)\\ (123)\\ (123)\\ \hline
\vdots\\ \hline
(12)\\ (12)\\ (12)\\ \vdots\\ (12)
\end{array}\right)
\in
\begin{array}{ccccc}
(123) & (123) & (13) & (13) & (13)\\
(123) & (13) & (123) & (13) & (13)\\
(123) & (13) & (13) & (123) & (13)\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
(13) & (13) & (13) & (123) & (123)\\ \hline
(123) & (13) & (13) & (13) & (13)\\
(13) & (123) & (13) & (13) & (13)\\
(13) & (13) & (123) & (13) & (13)\\
(13) & (13) & (13) & (123) & (13)\\
(13) & (13) & (13) & (13) & (123)\\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots\\ \hline
(12) & (12) & (23) & (23) & (23)\\
(12) & (23) & (12) & (23) & (23)\\
(12) & (23) & (23) & (12) & (23)\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
(23) & (23) & (23) & (12) & (12)
\end{array}
= R
$$

$$\mathbf{S}_3^{id}$$

Example: Group $\mathbf{S}_3$

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$$\mathbf{a} = \left(\begin{array}{c} (13)\\ (13)\\ (13)\\ \vdots\\ (13)\\ \hline (123)\\ (123)\\ (123)\\ (123)\\ (123)\\ \hline \vdots\\ \hline (12)\\ (12)\\ (12)\\ \vdots\\ (12) \end{array}\right) \in \begin{array}{ccccc} (123) & (123) & (13) & (13) & (13)\\ (123) & (13) & (123) & (13) & (13)\\ (123) & (13) & (13) & (123) & (13)\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ (13) & (13) & (13) & (123) & (123)\\ \hline (123) & (13) & (13) & (13) & (13)\\ (13) & (123) & (13) & (13) & (13)\\ (13) & (13) & (123) & (13) & (13)\\ (13) & (13) & (13) & (123) & (13)\\ (13) & (13) & (13) & (13) & (123)\\ \hline \vdots & \vdots & \vdots & \vdots & \vdots\\ \hline (12) & (12) & (23) & (23) & (23)\\ (12) & (23) & (12) & (23) & (23)\\ (12) & (23) & (23) & (12) & (23)\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ (23) & (23) & (23) & (12) & (12) \end{array} = R \qquad \mathbf{S}_3^{id}$$

$$\begin{array}{c} \{(13)\} \lll S_3\\ \{(13)\} \lll S_3\\ \{(13)\} \lll S_3\\ \vdots\\ \{(13)\} \lll S_3\\ \{(123)\} \lll S_3\\ \{(123)\} \lll S_3\\ \{(123)\} \lll S_3\\ \{(123)\} \lll S_3\\ \{(123)\} \lll S_3\\ \vdots\\ \{(12)\} \lll \{(12),(13),(23)\}\\ \{(12)\} \lll \{(12),(13),(23)\}\\ \{(12)\} \lll \{(12),(13),(23)\}\\ \vdots\\ \{(12)\} \lll \{(12),(13),(23)\} \end{array}$$

Example: Group $\mathbf{S}_3$

It is sufficient to show that

$$\mathbf{a} = \left(\begin{array}{c} (13) \\ (13) \\ (13) \\ \vdots \\ (13) \\ \hline (123) \\ (123) \\ (123) \\ (123) \\ (123) \\ \hline \vdots \\ \hline (12) \\ (12) \\ (12) \\ \vdots \\ (12) \end{array}\right) \in \begin{array}{ccccc} (123) & (123) & (13) & (13) & (13) \\ (123) & (13) & (123) & (13) & (13) \\ (123) & (13) & (13) & (123) & (13) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (13) & (13) & (13) & (123) & (123) \\ \hline (123) & (13) & (13) & (13) & (13) \\ (13) & (123) & (13) & (13) & (13) \\ (13) & (13) & (123) & (13) & (13) \\ (13) & (13) & (13) & (123) & (13) \\ (13) & (13) & (13) & (13) & (123) \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline (12) & (12) & (23) & (23) & (23) \\ (12) & (23) & (12) & (23) & (23) \\ (12) & (23) & (23) & (12) & (23) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (23) & (23) & (23) & (12) & (12) \end{array} = R \qquad \mathbf{S}_3^{id}$$

$$\begin{array}{c} \{(13)\} \lll S_3 \\ \{(13)\} \lll S_3 \\ \{(13)\} \lll S_3 \\ \vdots \\ \{(13)\} \lll S_3 \\ \{(123)\} \lll S_3 \\ \{(123)\} \lll S_3 \\ \{(123)\} \lll S_3 \\ \{(123)\} \lll S_3 \\ \{(123)\} \lll S_3 \\ \vdots \\ \{(12)\} \lll \{(12),(13),(23)\} \\ \{(12)\} \lll \{(12),(13),(23)\} \\ \{(12)\} \lll \{(12),(13),(23)\} \\ \vdots \\ \{(12)\} \lll \{(12),(13),(23)\} \end{array}$$

Assume $\mathbf{a} \notin R \leq_{sd} \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$.

Example: Group $\mathbf{S}_3$

It is sufficient to show that

$$\mathbf{a} = \begin{pmatrix} (13) \\ (13) \\ (13) \\ \vdots \\ (13) \\ \hline (123) \\ (123) \\ (123) \\ (123) \\ (123) \\ \hline \vdots \\ \hline (12) \\ (12) \\ (12) \\ \vdots \\ (12) \end{pmatrix} \in \begin{array}{ccccc} (123) & (123) & (13) & (13) & (13) \\ (123) & (13) & (123) & (13) & (13) \\ (123) & (13) & (13) & (123) & (13) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (13) & (13) & (13) & (123) & (123) \\ \hline (123) & (13) & (13) & (13) & (13) \\ (13) & (123) & (13) & (13) & (13) \\ (13) & (13) & (123) & (13) & (13) \\ (13) & (13) & (13) & (123) & (13) \\ (13) & (13) & (13) & (13) & (123) \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline (12) & (12) & (23) & (23) & (23) \\ (12) & (23) & (12) & (23) & (23) \\ (12) & (23) & (23) & (12) & (23) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (23) & (23) & (23) & (12) & (12) \end{array} = R \qquad \begin{array}{c} \{(13)\} \lll S_3 \\ \{(13)\} \lll S_3 \\ \{(13)\} \lll S_3 \\ \vdots \\ \{(13)\} \lll S_3 \\ \{(123)\} \lll S_3 \\ \{(123)\} \lll S_3 \\ \{(123)\} \lll S_3 \\ \{(123)\} \lll S_3 \\ \{(123)\} \lll S_3 \\ \vdots \\ \{(12)\} \lll \{(12),(13),(23)\} \\ \{(12)\} \lll \{(12),(13),(23)\} \\ \{(12)\} \lll \{(12),(13),(23)\} \\ \vdots \\ \{(12)\} \lll \{(12),(13),(23)\} \end{array}$$

$$\mathbf{S}_3^{id}$$

Assume $\mathbf{a} \notin R \leq_{sd} \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$. Choose a maximal $C_1 \lll \mathbf{A}_1, \ldots, C_n \lll \mathbf{A}_n$ such that $\mathbf{a} \in (C_1 \times \cdots \times C_n)$ and

$$R \cap (C_1 \times \cdots \times C_n) = \varnothing.$$

Example: Group $\mathbf{S}_3$

It is sufficient to show that

$$
\mathbf{a} = \left(\begin{array}{c}
(13)\\ (13)\\ (13)\\ \vdots\\ (13)\\ \hline
(123)\\ (123)\\ (123)\\ (123)\\ (123)\\ \hline
\vdots\\ \hline
(12)\\ (12)\\ (12)\\ \vdots\\ (12)
\end{array}\right)
\in
\begin{array}{ccccc}
(123) & (123) & (13) & (13) & (13)\\
(123) & (13) & (123) & (13) & (13)\\
(123) & (13) & (13) & (123) & (13)\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
(13) & (13) & (13) & (123) & (123)\\ \hline
(123) & (13) & (13) & (13) & (13)\\
(13) & (123) & (13) & (13) & (13)\\
(13) & (13) & (123) & (13) & (13)\\
(13) & (13) & (13) & (123) & (13)\\
(13) & (13) & (13) & (13) & (123)\\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots\\ \hline
(12) & (12) & (23) & (23) & (23)\\
(12) & (23) & (12) & (23) & (23)\\
(12) & (23) & (23) & (12) & (23)\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
(23) & (23) & (23) & (12) & (12)
\end{array}
= R
\qquad \mathbf{S}_3^{id}
\qquad
\begin{array}{c}
\{(13)\} \lll S_3\\
\{(13)\} \lll S_3\\
\{(13)\} \lll S_3\\
\vdots\\
\{(13)\} \lll S_3\\
\{(123)\} \lll S_3\\
\{(123)\} \lll S_3\\
\{(123)\} \lll S_3\\
\{(123)\} \lll S_3\\
\{(123)\} \lll S_3\\
\vdots\\
\{(12)\} \lll \{(12),(13),(23)\}\\
\{(12)\} \lll \{(12),(13),(23)\}\\
\{(12)\} \lll \{(12),(13),(23)\}\\
\vdots\\
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$$

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- For every i either $C_i = A_i$, or $|C_i| = 1$ and $A_i = S_3$;

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\end{array}\right)
\in
\begin{array}{ccccc}
(123) & (123) & (13) & (13) & (13)\\
(123) & (13) & (123) & (13) & (13)\\
(123) & (13) & (13) & (123) & (13)\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
(13) & (13) & (13) & (123) & (123)\\ \hline
(123) & (13) & (13) & (13) & (13)\\
(13) & (123) & (13) & (13) & (13)\\
(13) & (13) & (123) & (13) & (13)\\
(13) & (13) & (13) & (123) & (13)\\
(13) & (13) & (13) & (13) & (123)\\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots\\ \hline
(12) & (12) & (23) & (23) & (23)\\
(12) & (23) & (12) & (23) & (23)\\
(12) & (23) & (23) & (12) & (23)\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
(23) & (23) & (23) & (12) & (12)
\end{array}
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\qquad
\begin{array}{c}
\{(13)\} \lll S_3\\
\{(13)\} \lll S_3\\
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\vdots\\
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- For every i either $C_i = A_i$, or $|C_i| = 1$ and $A_i = S_3$;
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Example: Group S_3

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\in
\begin{array}{ccccc}
(123) & (123) & (13) & (13) & (13)\\
(123) & (13) & (123) & (13) & (13)\\
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\vdots & \vdots & \vdots & \vdots & \vdots\\
(13) & (13) & (13) & (123) & (123)\\ \hline
(123) & (13) & (13) & (13) & (13)\\
(13) & (123) & (13) & (13) & (13)\\
(13) & (13) & (123) & (13) & (13)\\
(13) & (13) & (13) & (123) & (13)\\
(13) & (13) & (13) & (13) & (123)\\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots\\ \hline
(12) & (12) & (23) & (23) & (23)\\
(12) & (23) & (12) & (23) & (23)\\
(12) & (23) & (23) & (12) & (23)\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
(23) & (23) & (23) & (12) & (12)
\end{array}
= R \quad \mathbf{S}_3^{id}
\qquad
\begin{array}{c}
\{(13)\} \lll S_3\\
\{(13)\} \lll S_3\\
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\vdots\\
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- $\mathrm{pr}_{i,j}(R)$ is not linked whenever $C_i \neq A_i$ and $C_j \neq A_j$;
- A contradiction should appear on a fixed shape e.g. (x, y, x, x, y).

Decomposition using bridges

Theorem

Decomposition using bridges

Theorem

Suppose

- $0_{\mathbf{A}}$ is $\wedge$-irreducible.

Decomposition using bridges

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- $0_{\mathbf{A}}$ is $\wedge$-irreducible.
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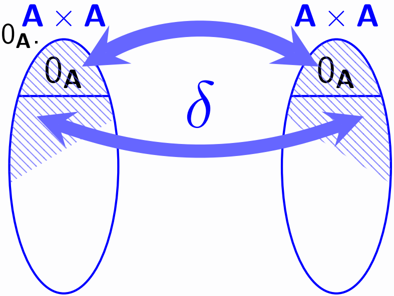
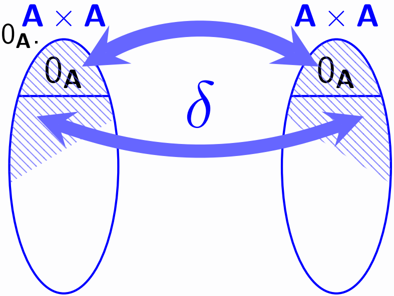


Decomposition using bridges

Theorem

Suppose

- $0_{\mathbf{A}}$ is $\wedge$-irreducible.
- $\delta \leq \mathbf{A}^4$ is a bridge from $0_{\mathbf{A}}$ to $0_{\mathbf{A}}$.
- $\mathrm{tr}(\delta) = A \times A$.



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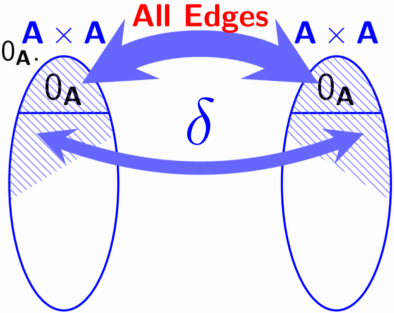

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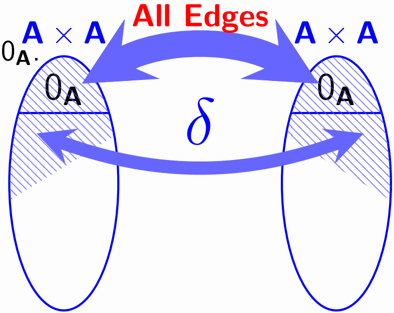
Then $\mathbf{A} \cong \mathbf{A}/0^*_{\mathbf{A}} \boxtimes \mathbf{G}$
for some affine $\mathbf{G}$



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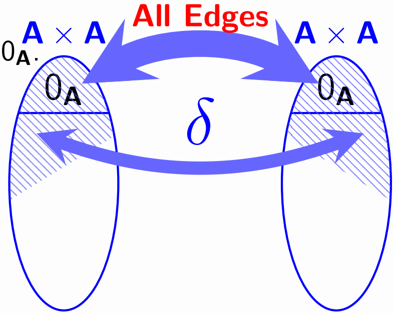

$\mathbf{A} \cong \mathbf{B} \boxtimes \mathbf{G}$:

- domain $A = B \times G$
- $f^{\mathbf{A}}(x_1, \ldots, x_n) = (f^{\mathbf{B}}(x_1^{(1)}, \ldots, x_n^{(1)}),$
  $d_f(x_1^{(1)}, \ldots, x_n^{(1)}) + r_1 x_1^{(2)} + r_2 x_2^{(2)} + \cdots + r_n x_n^{(2)})$

Results

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Existence of an XY-symmetric operation

Every finite Taylor algebra has XY-symmetric term operations.

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A simplified proof of the CSP Dichotomy Conjecture

12 pages modulo properties of strong subalgebras.

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Existence of palette symmetric operations

Every Taylor algebra $\mathbf{A} = \mathbf{B}_1 \times \cdots \times \mathbf{B}_n$, where $|B_i| < 8$ for every i has palette symmetric term operations.

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Every Taylor algebra $\mathbf{A} = \mathbf{B}_1 \times \cdots \times \mathbf{B}_n$, where $|B_i| < 8$ for every i has palette symmetric term operations.

CSP(Γ) is solvable by the singleton(BLP+AIP) algorithm for any tractable (multi-sorted) Γ on domains of size at most 7.

Thank you for your attention