On similarities in minimal Taylor algebras between objects considered by Zhuk and Bulatov in their Dichotomy proofs

Vlado Uljarević

Department of Mathematics and Informatics,
Faculty of Sciences, University of Novi Sad, Serbia

joint work with Zarathustra Brady, Gergő Gyenizse, Petar Đapić, Marcin Kozik, Petar Marković, and Aleksandar Prokić

AAA 108 — 108. Arbeitstagung Allgemeine Algebra
108th Workshop on General Algebra

6 – 8 February 2026, Vienna

Formulation of the CSP and the Dichotomy Conjecture

The Fixed-Template Constraint Satisfaction Problem

Let $\mathbb{A}$ be a finite relational structure in a finite signature. $CSP(\mathbb{A})$ is the following decision problem:

Given a finite structure $\mathbb{B}$ in the same signature as $\mathbb{A}$, decide whether or not there is a homomorphism from $\mathbb{B}$ into $\mathbb{A}$.

Formulation of the CSP and the Dichotomy Conjecture

The Fixed-Template Constraint Satisfaction Problem

Let $\mathbb{A}$ be a finite relational structure in a finite signature. $CSP(\mathbb{A})$ is the following decision problem:

Given a finite structure $\mathbb{B}$ in the same signature as $\mathbb{A}$, decide whether or not there is a homomorphism from $\mathbb{B}$ into $\mathbb{A}$.

The Dichotomy Conjecture (Feder, Vardi, 1993)

$CSP(\mathbb{A})$ is either solvable in polynomial time, or NP-complete.

Formulation of the CSP and the Dichotomy Conjecture

The Fixed-Template Constraint Satisfaction Problem

Let $\mathbb{A}$ be a finite relational structure in a finite signature. $CSP(\mathbb{A})$ is the following decision problem:

Given a finite structure $\mathbb{B}$ in the same signature as $\mathbb{A}$, decide whether or not there is a homomorphism from $\mathbb{B}$ into $\mathbb{A}$.

The Dichotomy Conjecture (Feder, Vardi, 1993)

$CSP(\mathbb{A})$ is either solvable in polynomial time, or NP-complete.

Algebraic Dichotomy Conjecture (Bulatov, Jeavons and Krokhin)

Let $\mathbb{A}$ be a finite relational structure in a finite signature. Then:

1. If $\mathbb{A}$ has a Taylor polymorphism, then $CSP(\mathbb{A})$ is in P;
2. Otherwise, $CSP(\mathbb{A})$ is NP-complete.

Formulation of the CSP and the Dichotomy Conjecture

The Fixed-Template Constraint Satisfaction Problem

Let $\mathbb{A}$ be a finite relational structure in a finite signature. $CSP(\mathbb{A})$ is the following decision problem:

Given a finite structure $\mathbb{B}$ in the same signature as $\mathbb{A}$, decide whether or not there is a homomorphism from $\mathbb{B}$ into $\mathbb{A}$.

The Dichotomy Conjecture (Feder, Vardi, 1993)

$CSP(\mathbb{A})$ is either solvable in polynomial time, or NP-complete.

Dichotomy Theorem (Bulatov; Zhuk, 2017)

Let $\mathbb{A}$ be a finite relational structure in a finite signature. Then:

1. If $\mathbb{A}$ has a Taylor polymorphism, then $CSP(\mathbb{A})$ is in P;
2. Otherwise, $CSP(\mathbb{A})$ is NP-complete.

Minimal Taylor Algebras

Both Dichotomy proofs use term reducts of the original Taylor algebra.

Minimal Taylor Algebras

Both Dichotomy proofs use term reducts of the original Taylor algebra.

Any finite Taylor algebra $\mathbf{A}$ has a term reduct $\mathbf{A}'$ which is still a Taylor algebra, but no proper subclone of $\mathrm{Clo}\,\mathbf{A}'$ has a Taylor operation.

Minimal Taylor Algebras

Both Dichotomy proofs use term reducts of the original Taylor algebra.

Any finite Taylor algebra $\mathbf{A}$ has a term reduct $\mathbf{A}'$ which is still a Taylor algebra, but no proper subclone of $\mathrm{Clo}\,\mathbf{A}'$ has a Taylor operation.

Such algebras are **minimal Taylor algebras**, for short **MTA**s, and any of their Taylor terms generates the whole clone.

Minimal Taylor Algebras

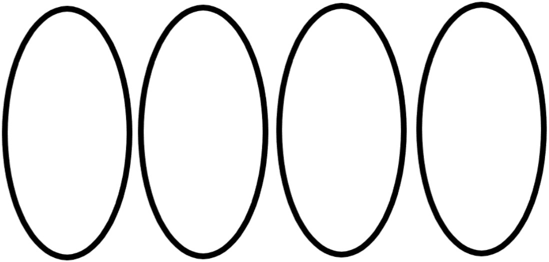
Both Dichotomy proofs use term reducts of the original Taylor algebra.

Any finite Taylor algebra $\mathbf{A}$ has a term reduct $\mathbf{A}'$ which is still a Taylor algebra, but no proper subclone of $\operatorname{Clo} \mathbf{A}'$ has a Taylor operation.

Such algebras are **minimal Taylor algebras**, for short **MTA**s, and any of their Taylor terms generates the whole clone.

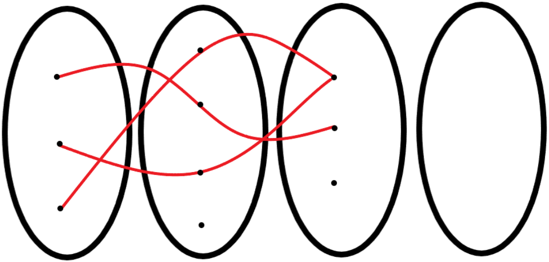
Any finite MTA has a ternary operation which generates its clone.

How to think of an instance



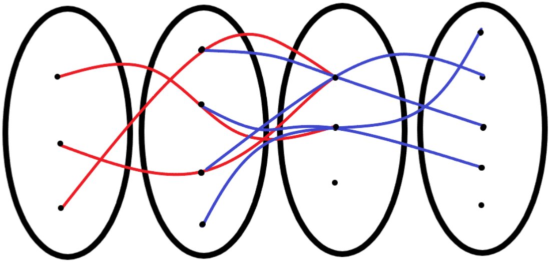
For each element of B there is the set of possible values, initially all of A.

How to think of an instance



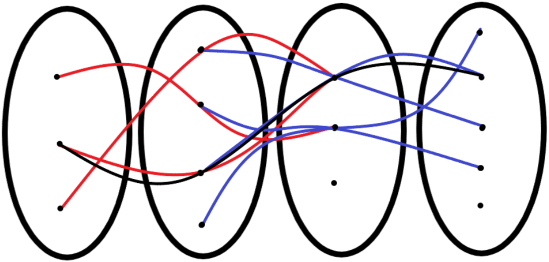
A constraint shows all the ways in which a homomorphism from $\mathbb{B}$ to $\mathbb{A}$ can restrict to some subset.

How to think of an instance



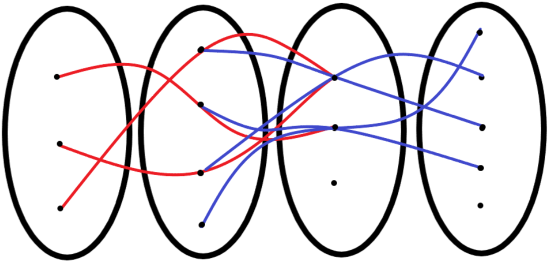
Of course, there are several constraints to satisfy.

How to think of an instance



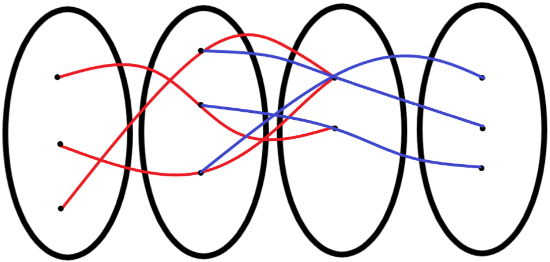
A solution which satisfies both the red and the blue constraint is depicted by the black line.

Consistency checking I: local consistency



Returning to the original setup.

Consistency checking I: local consistency



Some tuples can be removed from a constraint relation. We can also remove from the domains of variables the points that no tuple passes through.

Consistency checking I: local consistency

We can, in polynomial time, reduce our instance to an equivalent one such that:

Consistency checking I: local consistency

We can, in polynomial time, reduce our instance to an equivalent one such that:

- For every subset $S \subseteq B$, $|S| \leq 2$, and any pair of constraints (S_1, R_1) and (S_2, R_2) such that $S \subseteq S_1, S_2$, the restrictions of R_1 and of R_2 to S are equal (**2-consistency**),

Consistency checking I: local consistency

We can, in polynomial time, reduce our instance to an equivalent one such that:

- For every subset $S \subseteq B$, $|S| \leq 2$, and any pair of constraints (S_1, R_1) and (S_2, R_2) such that $S \subseteq S_1, S_2$, the restrictions of R_1 and of R_2 to S are equal (**2-consistency**), and
- For every triple $b_1, b_2, b_3 \in B$, there exists a constraint (S, R) such that $\{b_1, b_2, b_3\} \subseteq S$ (**3-density**).

Consistency checking I: local consistency

We can, in polynomial time, reduce our instance to an equivalent one such that:

- For every subset $S \subseteq B$, $|S| \leq 2$, and any pair of constraints (S_1, R_1) and (S_2, R_2) such that $S \subseteq S_1, S_2$, the restrictions of R_1 and of R_2 to S are equal (**2-consistency**), and
- For every triple $b_1, b_2, b_3 \in B$, there exists a constraint (S, R) such that $\{b_1, b_2, b_3\} \subseteq S$ (**3-density**).

(To ensure the 3-density, we first add the constraints of the form $(\{b_1, b_2, b_3\}, A \times A \times A)$ for any b_1, b_2, b_3 that fail the 3-density condition.)

Consistency checking I: local consistency

We can, in polynomial time, reduce our instance to an equivalent one such that:

- For every subset $S \subseteq B$, $|S| \leq 2$, and any pair of constraints (S_1, R_1) and (S_2, R_2) such that $S \subseteq S_1, S_2$, the restrictions of R_1 and of R_2 to S are equal (**2-consistency**), and
- For every triple $b_1, b_2, b_3 \in B$, there exists a constraint (S, R) such that $\{b_1, b_2, b_3\} \subseteq S$ (**3-density**).

(To ensure the 3-density, we first add the constraints of the form $(\{b_1, b_2, b_3\}, A \times A \times A)$ for any b_1, b_2, b_3 that fail the 3-density condition.)

The CSP instance that satisfies the above requirements is **(2,3)-minimal**.

Consistency checking II: structural consistency

Various subinstances can be made by:

Consistency checking II: structural consistency

Various subinstances can be made by:

- restricting the constraints to some subset of B;

Consistency checking II: structural consistency

Various subinstances can be made by:

- restricting the constraints to some subset of B;
- considering a subset of constraints.

Consistency checking II: structural consistency

Various subinstances can be made by:

- restricting the constraints to some subset of B;
- considering a subset of constraints.

Zhuk makes subinstances by using both.

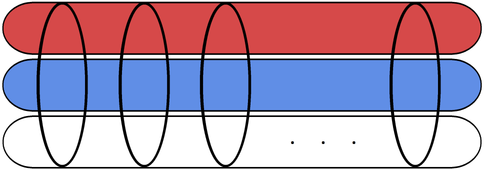
Consistency checking II: structural consistency

Various subinstances can be made by:

- restricting the constraints to some subset of B;
- considering a subset of constraints.

Zhuk makes subinstances by using both.

When all remaining variables in a subinstance are connected by the remaining constraints and the subinstance looks like this



Zhuk's irreducibility: the subinstance can be solved through any point.

Consistency checking II: structural consistency

Bulatov uses just subinstances obtained by restricting to subsets of variables.

Consistency checking II: structural consistency

Bulatov uses just subinstances obtained by restricting to subsets of variables.

Bulatov's block-minimality: we can solve such subinstances through any tuple in any constraint relation.

Consistency checking II: structural consistency

Bulatov uses just subinstances obtained by restricting to subsets of variables.

Bulatov's block-minimality: we can solve such subinstances through any tuple in any constraint relation.

Theorem (Zhuk, Bulatov)

Let P be a CSP instance. P can be converted in polynomial time to an equivalent one which is either Zhuk irreducible, or block-minimal.

Consistency checking II: structural consistency

Bulatov uses just subinstances obtained by restricting to subsets of variables.

Bulatov's block-minimality: we can solve such subinstances through any tuple in any constraint relation.

Theorem (Zhuk, Bulatov)

Let P be a CSP instance. P can be converted in polynomial time to an equivalent one which is either Zhuk irreducible, or block-minimal.

Proposition

Any $(2, 3)$-minimal instance which is block-minimal must also be Zhuk irreducible.

Zhuk's four types theorem

Theorem (Zhuk)

Let **A** be a MTA, then:

1. **A** has a nontrivial 2-absorbing subuniverse, or
2. **A** has a nontrivial center, or
3. $\mathbf{A}/\theta$ is affine for some proper congruence θ, or
4. $\mathbf{A}/\theta$ is polynomially complete for some proper congruence θ.

Zhuk's four types theorem

Theorem (Zhuk)

Let **A** be a MTA, then:

1. **A** has a nontrivial 2-absorbing subuniverse, or
2. **A** has a nontrivial center, or
3. $\mathbf{A}/\theta$ is affine for some proper congruence θ, or
4. $\mathbf{A}/\theta$ is polynomially complete for some proper congruence θ.

Theorem (The Zhuk reduction)

Let be P a consistent enough *CSP* instance and **A** a minimal Taylor algebra.

- If $\mathbf{C} \leq \mathbf{A}$ is either a binary absorbing subuniverse, or a center, or
- **A** has no binary absorbing subuniverse, nor a center, $\mathbf{A}/\theta$ is polynomially complete and C is a θ-block,

then P has a solution iff there exists a solution through some point of C.

Centers and $\rightarrow_{as}$

Bulatov gives three types of directed edges on any MTA: *a*-, *s*- and *m*-edges.

Centers and $\rightarrow_{as}$

Bulatov gives three types of directed edges on any MTA: *a*-, *s*- and *m*-edges.

$as(\mathbf{A})$ – the digraph that is the union of indicated edges.

Centers and $\rightarrow_{as}$

Bulatov gives three types of directed edges on any MTA: *a*-, *s*- and *m*-edges.

$as(\mathbf{A})$ – the digraph that is the union of indicated edges.

Definition (Zhuk)

A subset $C \subseteq A$ is a **center** of **A** if there exist a algebra **B** which has no binary absorbing subuniverses, and $R \leq_{sd} \mathbf{A} \times \mathbf{B}$ such that $C = \{a \in A : [a]R = B\}$. ($[a]R$:= the set of all right R-neighbors of a. Dual notion: $R[b]$.)

Centers and $\to_{as}$

Bulatov gives three types of directed edges on any MTA: *a*-, *s*- and *m*-edges.

$as(\mathbf{A})$ – the digraph that is the union of indicated edges.

Definition (Zhuk)

A subset $C \subseteq A$ is a **center** of **A** if there exist a algebra **B** which has no binary absorbing subuniverses, and $R \leq_{sd} \mathbf{A} \times \mathbf{B}$ such that $C = \{a \in A : [a]R = B\}$. ($[a]R$:= the set of all right R-neighbors of a. Dual notion: $R[b]$.)

Theorem (Barto, Brady, Bulatov, Kozik, Zhuk)

Let **A** be a MTA. $B \subseteq A$ is a center of **A** iff it is a ternary absorbing subset (=subuniverse) of **A** (written $\mathbf{B} \lhd_3 \mathbf{A}$).

Centers and $\to_{as}$

Bulatov gives three types of directed edges on any MTA: *a*-, *s*- and *m*-edges.

$as(\mathbf{A})$ – the digraph that is the union of indicated edges.

Definition (Zhuk)

A subset $C \subseteq A$ is a **center** of **A** if there exist a algebra **B** which has no binary absorbing subuniverses, and $R \leq_{sd} \mathbf{A} \times \mathbf{B}$ such that $C = \{a \in A : [a]R = B\}$. ($[a]R$:= the set of all right R-neighbors of a. Dual notion: $R[b]$.)

Theorem (Barto, Brady, Bulatov, Kozik, Zhuk)

Let **A** be a MTA. $B \subseteq A$ is a center of **A** iff it is a ternary absorbing subset (=subuniverse) of **A** (written $\mathbf{B} \lhd_3 \mathbf{A}$).

Theorem (Barto, Brady, Bulatov, Kozik, Zhuk)

Let **A** be a MTA. If $B \subseteq A$ is a center, $a \in B$ and $a \to_{as} b$, then $b \in B$.

Centers and $\to_{as}$

Bulatov gives three types of directed edges on any MTA: *a*-, *s*- and *m*-edges.

$as(\mathbf{A})$ – the digraph that is the union of indicated edges.

Definition (Zhuk)

A subset $C \subseteq A$ is a **center** of **A** if there exist a algebra **B** which has no binary absorbing subuniverses, and $R \leq_{sd} \mathbf{A} \times \mathbf{B}$ such that $C = \{a \in A : [a]R = B\}$. ($[a]R$:= the set of all right R-neighbors of a. Dual notion: $R[b]$.)

Theorem (Barto, Brady, Bulatov, Kozik, Zhuk)

Let **A** be a MTA. $B \subseteq A$ is a center of **A** iff it is a ternary absorbing subset (=subuniverse) of **A** (written $\mathbf{B} \lhd_3 \mathbf{A}$).

Theorem (Barto, Brady, Bulatov, Kozik, Zhuk)

Let **A** be a MTA. If $B \subseteq A$ is a center, $a \in B$ and $a \to_{as} b$, then $b \in B$.

Consequently, any center is the union of one or more strong as-components which form an order ideal in the poset of strong as-components.

Comparing centers and sink strong as-components I

Bulatov uses the sinks in the poset of strong components of $as(\mathbf{A})$.

Comparing centers and sink strong as-components I

Bulatov uses the sinks in the poset of strong components of $as(\mathbf{A})$.

Advantages of centers:

- Centers are subuniverses,

Comparing centers and sink strong as-components I

Bulatov uses the sinks in the poset of strong components of $as(\mathbf{A})$.

Advantages of centers:

- Centers are subuniverses,
- Centers are ternary absorbing and

Comparing centers and sink strong as-components I

Bulatov uses the sinks in the poset of strong components of $as(\mathbf{A})$.

Advantages of centers:

- Centers are subuniverses,
- Centers are ternary absorbing and
- Zhuk can reduce the domain of a variable in a (locally and structurally consistent enough) instance to the center of that domain.

Comparing centers and sink strong as-components II

If **A** is a MTA, denote $S(\mathbf{A}) := \{\text{all sink strong as-components of } \mathbf{A}\}$.

Comparing centers and sink strong as-components II

If **A** is a MTA, denote $S(\mathbf{A}) := \{$all sink strong as-components of $\mathbf{A}\}$.

Theorem (Bulatov's Maximality Lemma)

Let $\mathbf{R} \leq_{sd} \mathbf{A} \times \mathbf{B}$. If $R' \in S(\mathbf{R})$, then $pr_1\, R' \in S(\mathbf{A})$. If $A' \in S(\mathbf{A})$, then there exists $R' \in S(\mathbf{R})$ such that $pr_1\, R' = A'$.

Comparing centers and sink strong as-components II

If **A** is a MTA, denote $S(\mathbf{A}) := \{\text{all sink strong as-components of } \mathbf{A}\}$.

Theorem (Bulatov's Maximality Lemma)

Let $\mathbf{R} \leq_{sd} \mathbf{A} \times \mathbf{B}$. If $R' \in S(\mathbf{R})$, then $pr_1\, R' \in S(\mathbf{A})$. If $A' \in S(\mathbf{A})$, then there exists $R' \in S(\mathbf{R})$ such that $pr_1\, R' = A'$.

Theorem (Bulatov's Rectangularity Theorem)

Let $\mathbf{R} \leq_{sd} \mathbf{A} \times \mathbf{B}$ be linked. If $A' \in S(\mathbf{A})$, $B' \in S(\mathbf{B})$ and $R \cap A' \times B' \neq \emptyset$, then $A' \times B' \subseteq R$.

Comparing centers and sink strong as-components II

If $\mathbf{A}$ is a MTA, denote $S(\mathbf{A}) := \{$all sink strong as-components of $\mathbf{A}\}$.

Theorem (Bulatov's Maximality Lemma)

Let $\mathbf{R} \leq_{sd} \mathbf{A} \times \mathbf{B}$. If $R' \in S(\mathbf{R})$, then $pr_1\ R' \in S(\mathbf{A})$. If $A' \in S(\mathbf{A})$, then there exists $R' \in S(\mathbf{R})$ such that $pr_1\ R' = A'$.

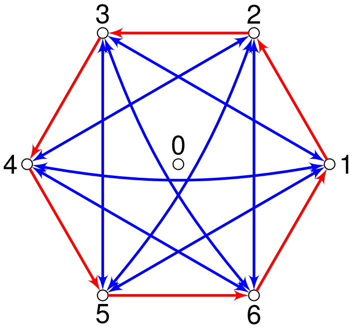
Theorem (Bulatov's Rectangularity Theorem)

Let $\mathbf{R} \leq_{sd} \mathbf{A} \times \mathbf{B}$ be linked. If $A' \in S(\mathbf{A})$, $B' \in S(\mathbf{B})$ and $R \cap A' \times B' \neq \emptyset$, then $A' \times B' \subseteq R$.

Theorem (Bulatov's Quasi-2-Decomposability Theorem)

Let $\mathbf{R} \leq_{sd} \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$, $\mathbf{a} \in A_1 \times \cdots \times A_n$ and for all $i, j \leqslant n$ let $R'_{ij} \in S(pr_{ij}R)$. If $(\mathbf{a}(i), \mathbf{a}(j)) \in R'_{ij}$ for all $i, j \leqslant n$, then there exists $R' \in S(\mathbf{R})$ such that, for all $i, j \leqslant n$, $pr_{ij}R' = R'_{ij}$.

7-element MTA with a sink strong as-component which is not a subuniverse

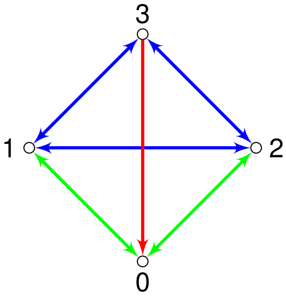


(Blue edges are m-edges, s-edges are colored red.)

The algebra $\mathbf{A}_1$ has the ternary symmetric operation t :

- A permutation $\varphi = (0)(123456)$ is an automorphism on $\mathbf{A}_1$.
- Each $\{0, x\}, 0 \neq x$, is the universe of a two-element majority subalgebra.
- $(\{1,2,3\}; t) \cong \mathbf{T}_1^C$, i.e. $t(1,2,3) = 3$.
- $(\{1,2,x\}; t) \cong \mathbf{T}_4^C, x \in \{0,4,5\}$, i.e. $t(1,2,4) = 2$.
- $(\{0,1,3,5\}; t) \cong \mathbf{M}_1$ (T. Waldhauser), i.e. $t(x,y,z) = 0$ if $\{x,y,z\} \subseteq \{0,1,3,5\}$ and $|\{x,y,z\}| = 3$.

4-element centerless MTA with comparable as-components



The algebra $\mathbf{A}_2$ has the ternary symmetric operation t :

- $(\{1,2,3\};t) \cong \mathbf{T}^C_{14}$, i.e. $t(1,2,3) = 3$.
- $(\{0,1,2\};t) \cong \mathbf{T}^C_{12}$, i.e. $t(0,1,2) = 0$.
- $(\{0,1,3\};t) \cong (\{0,2,3\};t) \cong \mathbf{T}^C_{10}$, i.e. $t(0,1,3) = 1$ and $t(0,2,3) = 2$.

(Blue edges represent m-edges, a-edges are green, while s-edges are colored red.)

Centers vs sink strong as-components

Useful properties which hold in centers and those that hold in sink strong as-components:

	Sink as-comp	Centers
Subuniverse	✗	✓
$\lhd_3$	✗	✓
Reduce domains of variables	?	✓
Maximality	✓	?
Rectangularity	✓	?
Quasi-2-Decomposability	✓	?

Centers vs sink strong as-components

If we denote by $C(\mathbf{A})$ the set of all centers of a MTA $\mathbf{A}$, the version of the Maximality Theorem in which we replace $S(\mathbf{A})$ by $C(\mathbf{A})$ holds.

	Sink as-comp	Centers
Subuniverse	✗	✓
$\lhd_3$	✗	✓
Reduce domains of variables	?	✓
Maximality	✓	✓
Rectangularity	✓	?
Quasi-2-Decomposability	✓	?

Centers vs sink strong as-components

However, the Rectangularity Theorem for centers fails,

	Sink as-comp	Centers
Subuniverse	✗	✓
$\lhd_3$	✗	✓
Reduce domains of variables	?	✓
Maximality	✓	✓
Rectangularity	✓	✗
Quasi-2-Decomposability	✓	?

Centers vs sink strong as-components

However, the Rectangularity Theorem for centers fails, and we also expect the Quasi-2-Decomposability Theorem to fail for centers.

	Sink as-comp	Centers
Subuniverse	✗	✓
$\lhd_3$	✗	✓
Reduce domains of variables	?	✓
Maximality	✓	✓
Rectangularity	✓	✗
Quasi-2-Decomposability	✓	?

Centers vs sink strong as-components

It seems that we need to decide whether we prefer Zhuk's centers or Bulatov's sink strong as-components.

	Sink as-comp	Centers
Subuniverse	✗	✓
$\lhd_3$	✗	✓
Reduce domains of variables	?	✓
Maximality	✓	✓
Rectangularity	✓	✗
Quasi-2-Decomposability	✓	?

Centers vs sink strong as-components

If we replace the centers with minimal centers (under inclusion), everything holds.

	Sink as-comp	Centers	Min. centers
Subuniverse	✗	✓	✓
$\lhd_3$	✗	✓	✓
Reduce domains of variables	?	✓	✓
Maximality	✓	✓	✓
Rectangularity	✓	✗	✓
Quasi-2-Decomposability	✓	?	✓

What we did about bridges and inseparability

In Zhuk's Dichotomy proof, a key notion is that of a bridge between a congruence of $\mathbf{A}_i$ and a congruence of $\mathbf{A}_j$. The bridge is a 4-ary compatible relation $\delta \leq \mathbf{A}_i \times \mathbf{A}_i \times \mathbf{A}_j \times \mathbf{A}_j$, which naturally defines the binary relation $\tilde{\delta} := \delta(x, x, y, y)$.

What we did about bridges and inseparability

In Zhuk's Dichotomy proof, a key notion is that of a bridge between a congruence of $\mathbf{A}_i$ and a congruence of $\mathbf{A}_j$. The bridge is a 4-ary compatible relation $\delta \leq \mathbf{A}_i \times \mathbf{A}_i \times \mathbf{A}_j \times \mathbf{A}_j$, which naturally defines the binary relation $\tilde{\delta} := \delta(x, x, y, y)$.

In Bulatov's papers there is a notion of inseparability of a covering pair of congruences of $\mathbf{A}_i$ from a covering pair of congruences of $\mathbf{A}_j$.

What we did about bridges and inseparability

In Zhuk's Dichotomy proof, a key notion is that of a bridge between a congruence of $\mathbf{A}_i$ and a congruence of $\mathbf{A}_j$. The bridge is a 4-ary compatible relation $\delta \leq \mathbf{A}_i \times \mathbf{A}_i \times \mathbf{A}_j \times \mathbf{A}_j$, which naturally defines the binary relation $\tilde{\delta} := \delta(x, x, y, y)$.

In Bulatov's papers there is a notion of inseparability of a covering pair of congruences of $\mathbf{A}_i$ from a covering pair of congruences of $\mathbf{A}_j$. To connect to Zhuk's bridges, we consider inseparability with respect to the binary relation coming from the bridge, while Bulatov considers the inseparability with respect to the constraint relations.

What we did about bridges and inseparability

In Zhuk's Dichotomy proof, a key notion is that of a bridge between a congruence of $\mathbf{A}_i$ and a congruence of $\mathbf{A}_j$. The bridge is a 4-ary compatible relation $\delta \leq \mathbf{A}_i \times \mathbf{A}_i \times \mathbf{A}_j \times \mathbf{A}_j$, which naturally defines the binary relation $\tilde{\delta} := \delta(x, x, y, y)$.

In Bulatov's papers there is a notion of inseparability of a covering pair of congruences of $\mathbf{A}_i$ from a covering pair of congruences of $\mathbf{A}_j$. To connect to Zhuk's bridges, we consider inseparability with respect to the binary relation coming from the bridge, while Bulatov considers the inseparability with respect to the constraint relations. We proved:

1. If there exists a bridge δ between $\alpha \in \operatorname{Con} \mathbf{A}_i$ and $\beta \in \operatorname{Con} \mathbf{A}_j$, then there exist $\alpha_1, \alpha_2 \in \operatorname{Con} \mathbf{A}_i$ and $\beta_1, \beta_2 \in \operatorname{Con} \mathbf{A}_j$ such that $\alpha \leq \alpha_1 \prec \alpha_2$, $\beta \leq \beta_1 \prec \beta_2$ and (α_1, α_2) is mutually inseparable from (β_1, β_2) via $\tilde{\delta}$.
2. If $\alpha_1 \prec \alpha_2 \in \operatorname{Con} \mathbf{A}_i$ and $\beta_1 \prec \beta_2 \in \operatorname{Con} \mathbf{A}_j$ are such that (α_1, α_2) is mutually inseparable from (β_1, β_2) via some $R \leq_{sd} \mathbf{A} \times \mathbf{B}$, then there exists a bridge δ between α_1 and β_1. (We can not ensure $\tilde{\delta} = R$, at least for now.)

Bridges definition

Let $\mathbf{A}_1, \mathbf{A}_2$ be MTAs and $\alpha_i \in \operatorname{Con} \mathbf{A}_i$, $i = 1, 2$. We say that $\delta \leq \mathbf{A}_1 \times \mathbf{A}_1 \times \mathbf{A}_2 \times \mathbf{A}_2$ is a **bridge** between α_1 and α_2 if

- If $(a, b, c, d) \in \delta$, $(a', a), (b', b) \in \alpha_1$ and $(c', c), (d', d) \in \alpha_2$, then $(a', b', c', d') \in \delta$.
- For all $(a, b, c, d) \in \delta$, $(a, b) \in \alpha_1 \Leftrightarrow (c, d) \in \alpha_2$.
- For any $(a, b) \in \alpha_1$ there exist $(c, d) \in \alpha_2$ such that $(a, b, c, d) \in \delta$ and symmetrically, for any $(c, d) \in \alpha_2$ there exist $(a, b) \in \alpha_1$ such that $(a, b, c, d) \in \delta$.
- There exists $(a, b, c, d) \in \delta$ such that $(a, b) \notin \alpha_1$ (and therefore $(c, d) \notin \alpha_2$)

The bridge δ between α_1 and α_2 naturally defines the relation $\tilde{\delta} \leq_{sd} (\mathbf{A}_1/\alpha_1) \times (\mathbf{A}_2/\alpha_2)$ by $([a]_{\alpha_1}, [b]_{\alpha_2}) \in \tilde{\delta}$ iff $(a, a, b, b) \in \delta$.