$\sqcap$-ideals and $\sqcup$-filters in double Boolean algebras

By:
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February 6, 2026

Context and motivations

Double Boolean algebras are algebras $\underline{D} := (D; \sqcap, \sqcup, \neg, \lrcorner, \bot, \top)$ of type $(2, 2, 1, 1, 0, 0)$ introduced by Rudolf Wille to capture the equational theory of the algebra of protoconcepts.

Every double Boolean algebra $\underline{D}$ contains two Boolean algebras: $\underline{D}_{\sqcap}$ and $\underline{D}_{\sqcup}$.

As in classical algebraic structures, Wille defines the notion of an ideal (resp. a filter) in double Boolean algebra $\underline{D}$ as a subset I (resp. F) of D such that $\bot \in I$ (resp. $\top \in F$) and for all $x, y \in D$,

(1) $x, y \in I \Rightarrow x \sqcup y \in I$,

(2) $y \in I, x \sqsubseteq y \Rightarrow x \in I$.

(resp.

(1') $x, y \in F \Rightarrow x \sqcap y \in F$,

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In [5] Y. Tenkeu et al showed that the set of all ideals (resp. filters) of a dBa $\underline{D}$ is endowed with the structure of lattice isomorphic to the lattice of ideals (resp. filters) of the Boolean algebra $\underline{D}_{\sqcup}$ (resp. $\underline{D}_{\sqcap}$).

This result shows that the notion of ideal (resp. filter) defined by Wille allows us just to understand the Boolean algebra $\underline{D}_{\sqcup}$ (resp. $\underline{D}_{\sqcap}$) but not the dBa $\underline{D}$.

To overcome this, we introduce in this work a new class of ideals (resp. filters) in dBas called $\sqcap$-ideals ($\sqcup$-filters) and we study their properties.

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Preliminaries on double Boolean algebras

Definition 1

An algebra $\underline{D} = (D; \sqcap, \sqcup, \neg, \lrcorner, \bot, \top)$ *of type* $(2, 2, 1, 1, 0, 0)$ *is called* ***double Boolean algebra (dBa)*** *if the following axioms are satisfied:*

$$
\begin{array}{llll}
(1a) & (x \sqcap x) \sqcap y = x \sqcap y & (1b) & (x \sqcup x) \sqcup y = x \sqcup y \\
(2a) & x \sqcap y = y \sqcap x & (2b) & x \sqcup y = y \sqcup x \\
(3a) & x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z & (3b) & x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z \\
(4a) & x \sqcap (x \sqcup y) = x \sqcap x & (4b) & x \sqcup (x \sqcap y) = x \sqcup x \\
(5a) & x \sqcap (x \vee y) = x \sqcap x & (5b) & x \sqcup (x \wedge y) = x \sqcup x \\
(6a) & x \sqcap (y \vee z) = (x \sqcap y) \vee (x \sqcap z) & (6b) & x \sqcup (y \wedge z) = (x \sqcup y) \wedge (x \sqcup z) \\
(7a) & \neg\neg(x \sqcap y) = x \sqcap y & (7b) & \lrcorner\lrcorner(x \sqcup y) = x \sqcup y \\
(8a) & \neg(x \sqcap x) = \neg x & (8b) & \lrcorner(x \sqcup x) = \lrcorner x \\
(9a) & x \sqcap \neg x = \bot & (9b) & x \sqcup \lrcorner x = \top \\
(10a) & \neg\bot = \top \sqcap \top & (10b) & \lrcorner\top = \bot \sqcup \bot \\
(11a) & \neg\top = \bot & (11b) & \lrcorner\bot = \top
\end{array}
$$

$$(12)\ (x \sqcap x) \sqcup (x \sqcap x) = (x \sqcup x) \sqcap (x \sqcup x)$$

where $x \vee y = \neg(\neg x \sqcap \neg y)$ *and* $x \wedge y = \lrcorner(\lrcorner x \sqcup \lrcorner y)$

Preliminaries on double Boolean algebras

On a double Boolean algebra $\underline{D}$, a quasi-order relation $\sqsubseteq$ is defined as follows:

$$x \sqsubseteq y :\Longleftrightarrow x \sqcap y = x \sqcap x \text{ and } x \sqcup y = y \sqcup y. \quad (1)$$

- $D_{\sqcap} := \{x \in D \mid x \sqcap x = x\}$, $D_{\sqcup} := \{x \in D \mid x \sqcup x = x\}$, and $D_p := D_{\sqcap} \cup D_{\sqcup}$.

$\underline{D}_{\sqcap} = (D_{\sqcap}; \sqcap, \vee, \neg, \bot, \top \sqcap \top)$ and $\underline{D}_{\sqcup} = (D_{\sqcup}; \wedge, \sqcup, \lrcorner, \bot \sqcup \bot, \top)$ are Boolean algebras.

Definition 2 ([6, 2])

A dBa $\underline{D}$ is called :

1. ***Pure** if $D = D_p$.*
2. ***Trivial** if $\top \sqcap \top = \bot \sqcup \bot$.*
3. ***Regular** if "$\sqsubseteq$" is an order.*

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Example 1

The algebra $\underline{D}_3 := (\{\bot, a, \top\}; \sqcap, \sqcup, \neg, \lrcorner, \bot, \top)$ defined by the Hasse diagram and the Cayley tables in Fig. 1 is a **pure and trivial dBa**. Moreover $D_{3_\sqcap} = \{\bot, a\}$ and $D_{3_\sqcup} = \{a, \top\}$.

$\sqcap$	$\bot$	a	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	a
$\top$	$\bot$	a	a

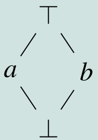
$\sqcup$	$\bot$	a	$\top$
$\bot$	a	a	$\top$
a	a	a	$\top$
$\top$	$\top$	$\top$	$\top$

x	$\bot$	a	$\top$
$\neg x$	a	$\bot$	$\bot$
$\lrcorner x$	$\top$	$\top$	a

Fig. 1: $\underline{D}_3$ and its Cayley tables

Example 2

The algebra $\underline{D}_4 := (\{\bot, a, b, \top\}; \sqcap, \sqcup, \neg, \lrcorner, \bot, \top)$ defined by the Hasse diagram and the Cayley tables in Fig. 2 is a **trivial dBa that is not pure**. Moreover $D_{4_\sqcap} = \{\bot, a\}$ and $D_{4_\sqcup} = \{a, \top\}$.



$\sqcap$	$\bot$	a	b	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	$\bot$	a
b	$\bot$	$\bot$	$\bot$	$\bot$
$\top$	$\bot$	a	$\bot$	a

$\sqcup$	$\bot$	a	b	$\top$
$\bot$	a	a	$\top$	$\top$
a	a	a	$\top$	$\top$
b	$\top$	$\top$	$\top$	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$

x	$\bot$	a	b	$\top$
$\neg x$	a	$\bot$	a	$\bot$
$\lrcorner x$	$\top$	$\top$	a	a

Fig.2: $\underline{D}_4$ and its Cayley tables

Preliminaries on double Boolean algebras

The following proposition is useful to perform calculations in dBas.

Proposition 1 ([3, 4])

Let $\underline{D}$ be a dBa. For all $x, y \in D$, the following statements hold.

(1) $x \sqcap y \in D_{\sqcap}$ *and* $x \sqcup y \in D_{\sqcup}$.

(2) $\neg x \in D_{\sqcap}$ *and* $\lrcorner x \in D_{\sqcup}$.

(3) $x \vee y \in D_{\sqcap}$ *and* $x \wedge y \in D_{\sqcup}$.

(4) $\neg(x \vee y) = \neg x \sqcap \neg y$ *and* $\neg(x \sqcap y) = \neg x \vee \neg y$.

(5) $\lrcorner(x \wedge y) = \lrcorner x \sqcup \lrcorner y$ *and* $\lrcorner(x \sqcup y) = \lrcorner x \wedge \lrcorner y$.

(6) $x \sqcap y \sqsubseteq x \vee y \sqsubseteq x \sqcup y$ *and* $x \sqcap y \sqsubseteq x \wedge y \sqsubseteq x \sqcup y$

The notion of $\sqcap$-ideal (resp. $\sqcup$-filter) in double Boolean algebras

Definition 3

Let $\underline{D}$ be a dBa. A subset I of D is called a $\sqcap$**-ideal** if $\perp \in I$ and for all $x, y \in D$

(i) $x, y \in I \Rightarrow x \vee y \in I$,

(ii) $y \in I \Rightarrow x \sqcap y \in I$.

Lemma 4

Let $\underline{D}$ be a dBa and $I \subseteq D$. Then the following statements are equivalent:

(1) $\forall x, y \in D, y \in I \Rightarrow x \sqcap y \in I$.

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From Lemma 4, we obtain the following characterization of $\sqcap$-ideals using the dBa quasi-order.

Proposition 2

Let $\underline{D}$ be a dBa and $I \subseteq D$. I is a $\sqcap$-ideal if and only if the following conditions are satisfied:

(i) $\perp \in I$,

(ii) $\forall\, x, y \in D,\ x, y \in I \Rightarrow x \vee y \in I$,

(iii) $\forall\, x, y \in D,\ y \in I$ *and* $x \sqsubseteq y \Rightarrow x \sqcap x \in I$.

The notion of $\sqcap$-ideal (resp. $\sqcup$-filter) in double Boolean algebras

The dual notion of $\sqcap$-ideal in dBa is the notion of $\sqcup$-filter:

Definition 5

Let $\underline{D}$ be a dBa. A subset F of D is called a **$\sqcup$-filter** if $\top \in F$ and for all $x, y \in D$,

(i) $x, y \in F \Rightarrow x \wedge y \in F$,

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The dual version of Proposition 2 is given by:

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The notion of $\sqcap$-ideal (resp. $\sqcup$-filter) in double Boolean algebras

We give some examples to illustrate our definition.

Example 6

We consider the dBa $\underline{D}_4$ of Example 2. Then the sets $I = \{\bot, b\}$ and $J = \{\bot, a, \top\}$ are $\sqcap$-ideals, but the set $M = \{\bot, b, \top\}$ is not a $\sqcap$-ideal (since $\top \in M$ but $\top \sqcap \top = a \notin M$).

The following lemma is a direct consequence of Definition 3 and Definition 5.

Lemma 7

Let $\underline{D}$ be a dBa. Then,

(1) *If I is a $\sqcap$-ideal, then $I \cap D_\sqcap$ is an ideal of the Boolean algebra $\underline{D}_\sqcap$.*

(2) *If F is a $\sqcup$-filter, then $F \cap D_\sqcup$ is a filter of the Boolean algebra $\underline{D}_\sqcup$.*

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The notion of $\sqcap$-ideal (resp. $\sqcup$-filter) in double Boolean algebras

Proposition 4

Let $\underline{D}$ be a dBa. Then,

(1) *Every ideal of the Boolean algebra $\underline{D}_{\sqcap}$ is a $\sqcap$-ideal.*

(2) *Every filter of the Boolean algebra $\underline{D}_{\sqcup}$ is a $\sqcup$-filter.*

Wille [6] defines ideals and filters in dBa as follows:

Definition 8 ([6])

Let $\underline{D}$ be a dBa. A subset I of D is called an **ideal** if $\bot \in I$ and for all $x, y \in D$,

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The notion of **filter** is defined dually.

The following proposition shows that the notion of $\sqcap$-ideal (resp. $\sqcup$-filter) in double Boolean algebras generalizes that of ideal (resp. filter) as defined by Rudolf Wille.

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In any dBa, ideals are $\sqcap$-ideals and filters are $\sqcup$-filters. The converse is not true.

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The structure of the set of $\sqcap$-ideals (resp. $\sqcup$-filters) in double Boolean algebras

We denote by $\mathcal{I}(\underline{D})$ (resp. $\mathcal{F}(\underline{D})$) the set of all ideals (resp. filter) of $\underline{D}$ and by $\mathcal{I}_{\sqcap}(\underline{D})$ (resp. $\mathcal{F}_{\sqcup}(\underline{D})$) the set of all $\sqcap$-ideals (resp. $\sqcup$-filters) of $\underline{D}$.
A **closure system** on a set A is a collection of subsets of A that contains A and closed under arbitrary intersections.

Lemma 9

Let $\underline{D}$ be a dBa. Then, the sets $\mathcal{I}_{\sqcap}(\underline{D})$ and $\mathcal{F}_{\sqcup}(\underline{D})$ are closure systems.

Notation 1

We denote by $I_{\sqcap}(X)$ the ***$\sqcap$-ideal generated*** *by X. If $X = \{a\}$, then $I_{\sqcap}(X)$ is denoted by $I_{\sqcap}(a)$ and is called the* ***principal $\sqcap$-ideal*** *generated by a. Dually, we denote by $F_{\sqcup}(X)$ the* ***$\sqcup$-filter generated*** *by X. If $X = \{a\}$, then $F_{\sqcup}(X)$ is denoted by $F_{\sqcup}(a)$ and is called the* ***principal $\sqcup$-filter*** *generated by a.*

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Theorem 10

Let $\underline{D}$ be a dBa.

1. *Then $\mathcal{I}_{\sqcap}(\underline{D})$ is a complete lattice in which sup and inf are given by:* $\bigwedge_{\sqcap\, i\in K} I_i = \bigcap_{i\in K} I_i$ *and* $\bigvee_{\sqcap\, i\in K} I_i = I_{\sqcap}\big(\bigcup_{i\in K} I_i\big)$
2. *Then $\mathcal{F}_{\sqcup}(\underline{D})$ is a complete lattice in which sup and inf are given by:* $\bigwedge_{\sqcup\, i\in K} G_i = \bigcap_{i\in K} G_i$ *and* $\bigvee_{\sqcup\, i\in K} G_i = F_{\sqcup}\big(\bigcup_{i\in K} G_i\big)$

The structure of the set of $\sqcap$-ideals (resp. $\sqcup$-filters) in double Boolean algebras

In the following proposition, we give a description of $I_\sqcap(X)$ and $F_\sqcup(X)$.

Proposition 6

Let $\underline{D}$ be a dBa, $a \in D$, $\emptyset \neq X \subseteq D$, $I_1, I_2 \in \mathcal{I}_\sqcap(\underline{D})$, $F_1, F_2 \in \mathcal{F}_\sqcup(\underline{D})$.

(1) $I_\sqcap(a) = \{a\} \cup \{x \in D_\sqcap \mid x \sqsubseteq a \vee a\}$

(2) $I_\sqcap(X) = X \cup \{x \in D_\sqcap \mid x \sqsubseteq x_1 \vee \cdots \vee x_n \textit{ for some } n \in \mathbb{N}^*, x_1, \cdots, x_n \in X\}$

(3) $I_1 \vee_\sqcap I_2 = I_\sqcap(I_1 \cup I_2) = I_1 \cup I_2 \cup \{x \in D_\sqcap \mid x \sqsubseteq x_1 \vee x_2 \textit{ for some } x_1 \in I_1, x_2 \in I_2\}$

(4) $F_\sqcup(a) = \{a\} \cup \{x \in D_\sqcup \mid a \wedge a \sqsubseteq x\}$

(5) $F_\sqcup(X) = X \cup \{x \in D_\sqcup \mid x_1 \wedge \cdots \wedge x_n \sqsubseteq x \textit{ for some } n \in \mathbb{N}^*, x_1, \cdots, x_n \in X\}$

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Theorem 11

Let $\underline{D}$ be a dBa. Then $\mathcal{I}_{\sqcap}(\underline{D})$ is an algebraic lattice where the compact elements are finitely generated $\sqcap$-ideals. Dually, $\mathcal{F}_{\sqcup}(\underline{D})$ is an algebraic lattice where the compact elements are finitely generated $\sqcup$-filters.

We denote by $\mathcal{I}_{\sqcap_p}(\underline{D})$ (resp. $\mathcal{F}_{\sqcup_p}(\underline{D})$) the set of all principal $\sqcap$-ideals (resp. $\sqcup$-filters) of $\underline{D}$.

Lemma 12

Let $\underline{D}$ be a dBa. The maps $\Psi : D \to \mathcal{I}_{\sqcap_p}(\underline{D})$, $a \mapsto I_{\sqcap}(a)$ and $\psi : D \to \mathcal{F}_{\sqcup_p}(\underline{D})$, $a \mapsto F_{\sqcup}(a)$ are bijections.

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Let $\underline{D}$ be a dBa. The maps $\Psi : D \to \mathcal{I}_{\sqcap p}(\underline{D})$, $a \mapsto I_{\sqcap}(a)$ and $\psi : D \to \mathcal{F}_{\sqcup p}(\underline{D})$, $a \mapsto F_{\sqcup}(a)$ are bijections.

The structure of the set of $\sqcap$-ideals (resp. $\sqcup$-filters) in double Boolean algebras

The bijections of Lemma 12 make possible to transfer the structure of double Boolean algebra of $\underline{D}$ to $\mathcal{I}_{\sqcap p}(\underline{D})$ and $\mathcal{F}_{\sqcup p}(\underline{D})$. Therefore we define:

- on $\mathcal{I}_{\sqcap p}(\underline{D})$ the operation $\sqcap, \sqcup, \neg, \lrcorner$ as follows:

$$I_\sqcap(a) \sqcap I_\sqcap(b) = I_\sqcap(a \sqcap b), \quad I_\sqcap(a) \sqcup I_\sqcap(b) = I_\sqcap(a \sqcup b)$$

$$\neg I_\sqcap(a) = I_\sqcap(\neg a), \quad \lrcorner I_\sqcap(a) = I_\sqcap(\lrcorner a)$$

- on $\mathcal{F}_{\sqcup p}(\underline{D})$ the operation $\sqcap, \sqcup, \neg, \lrcorner$ as follows:

$$F_\sqcup(a) \sqcap F_\sqcup(b) = F_\sqcup(a \sqcap b), \quad F_\sqcup(a) \sqcup F_\sqcup(b) = F_\sqcup(a \sqcup b)$$

$$\neg F_\sqcup(a) = F_\sqcup(\neg a), \quad \lrcorner F_\sqcup(a) = F_\sqcup(\lrcorner a)$$

The structure of the set of $\sqcap$-ideals (resp. $\sqcup$-filters) in double Boolean algebras

Theorem 13

Let $\underline{D}$ be a dBa. Then

(1) $\underline{\mathcal{I}_{\sqcap}}_p(\underline{D}) := (\mathcal{I}_{\sqcap p}(\underline{D}); \sqcap, \sqcup, \neg, \lrcorner, I_{\sqcap}(\bot), I_{\sqcap}(\top))$ *is a dBa isomorphic to* $\underline{D}$ *via the map* $\Psi\colon D \to \mathcal{I}_{\sqcap p}(\underline{D})$, $a \mapsto I_{\sqcap}(a)$.

(2) $\underline{\mathcal{F}_{\sqcup}}_p(\underline{D}) := (\mathcal{F}_{\sqcup p}(\underline{D}); \sqcap, \sqcup, \neg, \lrcorner, F_{\sqcup}(\bot), F_{\sqcup}(\top))$ *is a dBa isomorphic to* $\underline{D}$ *via the map* $\psi\colon D \to \mathcal{F}_{\sqcup p}(\underline{D})$, $a \mapsto F_{\sqcup}(a)$.

The structure of the set of ⊓-ideals (resp. ⊔-filters) in double Boolean algebras

An algebra $\underline{H} := (H; \wedge, \vee, \to, 0, 1)$ is called a **Heyting algebra** if $(H; \wedge, \vee, 0, 1)$ is a Bounded lattice and the following law of Residuation holds: for all $x, y, z \in H$,

$$z \wedge x \leq y \quad \text{if and only if} \quad z \leq x \to y.$$

For any $I_1, I_2 \in \mathcal{I}_{\sqcap}(D)$, $F_1, F_2 \in \mathcal{F}_{\sqcup}(D)$, we set :

$$I_1 \to I_2 = \{x \in D \mid I_{\sqcap}(x) \cap I_1 \subseteq I_2\} \text{ and } F_1 \to F_2 = \{x \in D \mid F_{\sqcup}(x) \cap F_1 \subseteq F_2\}.$$

The structure of the set of ⊓-ideals (resp. ⊔-filters) in double Boolean algebras

An algebra $\underline{H} := (H; \wedge, \vee, \rightarrow, 0, 1)$ is called a **Heyting algebra** if $(H; \wedge, \vee, 0, 1)$ is a Bounded lattice and the following law of Residuation holds: for all $x, y, z \in H$,

$$z \wedge x \leq y \quad \text{if and only if} \quad z \leq x \rightarrow y.$$

For any $I_1, I_2 \in \mathcal{I}_{\sqcap}(D)$, $F_1, F_2 \in \mathcal{F}_{\sqcup}(D)$, we set :

$I_1 \rightarrow I_2 = \{x \in D \mid I_{\sqcap}(x) \cap I_1 \subseteq I_2\}$ and $F_1 \rightarrow F_2 = \{x \in D \mid F_{\sqcup}(x) \cap F_1 \subseteq F_2\}$.

The structure of the set of $\sqcap$-ideals (resp. $\sqcup$-filters) in double Boolean algebras

Lemma 14

Let $\underline{D}$ be a dBa, $I_1, I_2, I \in \mathcal{I}_\sqcap(D)$, $F_1, F_2, F \in \mathcal{F}_\sqcup(D)$, then the following statements hold:

(1) $I_1 \to I_2$ *is a* $\sqcap$*-ideal and* $F_1 \to F_2$ *is a* $\sqcup$*-filter.*

(2) $I_1 \cap I \subseteq I_2$ *iff* $I \subseteq I_1 \to I_2$.

(3) $F_1 \cap F \subseteq F_2$ *iff* $F \subseteq F_1 \to F_2$.

Theorem 15

For any dBa $\underline{D}$, $(\mathcal{I}_\sqcap(\underline{D}); \cap, \vee_\sqcap, \to, \{\bot\}, D)$ and $(\mathcal{F}_\sqcup(\underline{D}); \cap, \vee_\sqcup, \to, \{\top\}, D)$ are Heyting algebras.

The structure of the set of $\sqcap$-ideals (resp. $\sqcup$-filters) in double Boolean algebras

Lemma 14

Let $\underline{D}$ be a dBa, $I_1, I_2, I \in \mathcal{I}_\sqcap(D)$, $F_1, F_2, F \in \mathcal{F}_\sqcup(D)$, then the following statements hold:

(1) $I_1 \to I_2$ *is a* $\sqcap$*-ideal and* $F_1 \to F_2$ *is a* $\sqcup$*-filter.*

(2) $I_1 \cap I \subseteq I_2$ *iff* $I \subseteq I_1 \to I_2$.

(3) $F_1 \cap F \subseteq F_2$ *iff* $F \subseteq F_1 \to F_2$.

Theorem 15

For any dBa $\underline{D}$, $(\mathcal{I}_\sqcap(\underline{D}); \cap, \vee_\sqcap, \to, \{\bot\}, D)$ and $(\mathcal{F}_\sqcup(\underline{D}); \cap, \vee_\sqcup, \to, \{\top\}, D)$ are Heyting algebras.

Conclusion and perspectives

In this work, we have introduced $\sqcap$-ideals and $\sqcup$-filters in dBas. We have shown that this class of ideals (resp. filters) forms an algebraic and residuated lattice that is actually a Heyting algebra.

We plan in our future work to use this notion of $\sqcap$-ideals ($\sqcup$-filters) to a better understanding of the variety of double Boolean algebras. The study of spectral theory using $\sqcap$-ideals and $\sqcup$-filters should be considered.

Conclusion and perspectives

In this work, we have introduced $\sqcap$-ideals and $\sqcup$-filters in dBas. We have shown that this class of ideals (resp. filters) forms an algebraic and residuated lattice that is actually a Heyting algebra.

We plan in our future work to use this notion of $\sqcap$-ideals ($\sqcup$-filters) to a better understanding of the variety of double Boolean algebras. The study of spectral theory using $\sqcap$-ideals and $\sqcup$-filters should be considered.

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ACKNOWLEDGEMENT

We acknowledge the financial support of

- Prof. Sankappanavar
- The BFH-W
- The organizers of AAA 108

who make it possible to attend this conference.