Some Properties of Uncountable Linear Orders and their Automorphism Groups

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Introduction

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Theorem (Brouwer)

For every natural number $n \geq 1$ and every pair $A, B \subseteq \mathbb{R}^n$ which are countable and dense there is a homeomorphism $h : \mathbb{R}^n \cong \mathbb{R}^n$ so that $h``A = B$.

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- The goal of this talk is to introduce the case of study of such for Cantor's theorem.
- The uncountable version of Cantor's theorem is known as Baumgartner's Axiom (BA) and is independent of ZFC.
- In the rest of the talk we will sketch the background on BA and its applications as well as some new work due to myself jointly with Marun and Shelah.

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• A little less trivial, consider a dense linear order which can be covered by countably many closed, nowhere dense subsets of $\mathbb{R}$ versus one that is the intersection of countably many dense subsets of $\mathbb{R}$ (not isomorphic by the Baire category theorem).

• In fact there are many (in fact $2^{2^{\aleph_0}}$-many) pairwise non-isomorphic subsets $A \subseteq \mathbb{R}$ of size $2^{\aleph_0}$.

• Also, consider linear orders with uncountable intersection in *every* open interval versus those that are uncountable in some bounded region but then countable outside of that.

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We will be interested in separable linear orders L, in which case L will be isomorphic to an $\aleph_1$-dense set of reals - i.e. one whose intersection with every non-empty open interval has size $\aleph_1$. In what follows we will reserve $\aleph_1$-dense for this type of linear order.

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Strengthening Baumgartner's result Avraham, Rubin and Shelah proved the following. First recall that given a distributive lattice K an involution is a map $* : K \to K$ which is antimonotone and such that $(x \vee y)^* = x^* \wedge y^*$.

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For any finite distributive lattice with involution, K it is consistent that the homogeneous $\aleph_1$-dense sets (up to isomorphsim) under embeddability (with $\emptyset$) form a distributive lattice isomorphic to K with reversability the involution.

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Here the reverse of a linear order L is formally the linear order L^* given by flipping the order. If $L \subseteq \mathbb{R}$ then $L^* \cong \{-a \mid a \in L\}$. Under Martin's Axiom the homogeneous $\aleph_1$-dense sets form a distributive finite lattice so in some sense this theorem is best possible.

Martin's Axiom

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• Recall that this states that for each partial order $\mathbb{P}$ either has an uncountable antichain or else for each family of $\aleph_1$ many maximal antichains there is a filter $G \subseteq \mathbb{P}$ simultaneously intersecting them all.
• The technicalities of this statement are not important here but note that it implies many of the consequences of BA we have seen. For instance $\mathsf{MA}_{\aleph_1}$ implies the failure of CH - apply $\mathsf{MA}_{\aleph_1}$ to the partial order of finite binary sequences under end extension.

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For this reason the following theorem is very notable.

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They actually give several proofs of this theorem. The first shows that $\mathrm{MA}_{\aleph_1}$ is consistent with an $\aleph_1$-dense order L which is called essentially increasing: if $f : L \to L$ is a function with uncountable domain then it has an uncountable subset that is monotonically increasing. Note that this implies in particular that L is not isomorphic to its reverse $L^* = \{-a \mid a \in L\}$.

Martin's Axiom

In fact they show a little more - call an $\aleph_1$-dense set $L = \{a_\xi \mid \xi \in \omega_1\}$ good if for each $n < \omega$ and each family of disjoint, increasing n-tuples $\{\bar{b}_\xi \in [L]^n \mid \xi \in \omega_1\}$ there are $\xi < \eta$ so that for all $i < n$ $\bar{b}(i)_\xi < \bar{b}_\eta(i)$. It is not hard to check that under CH there are good $\aleph_1$-dense sets.

Lemma (Avraham-Shelah, '81)

$\mathrm{MA}_{\aleph_1}$ *is consistent with a good set. If L is good and* $\mathrm{MA}_{\aleph_1}$ *holds then L is essentially increasing.*

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If L is good then $\mathsf{MA}_{\aleph_1}$ implies that L is slicewise coverable.

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Theorem (Marun-Shelah-S.)

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Here L is slicewise coverable if given any partition of L into ω_1-many countable dense sets $\{L_\alpha \mid \alpha \in \omega_1\}$ there are countably many increasing functions $f_n : L \to L$ so that $\bigcup f_n = \bigcup_{\alpha \in \omega_1} L_\alpha \times L_\alpha$.

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• The point here is that if L is slicewise coverable then it cannot be made isomorphic to its reverse by an $\aleph_1$-sized forcing which preserves $\aleph_1$, thus strengthening the conclusion of the original Avraham-Shelah theorem.

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Let $f : L \to L$ have uncountable domain. One can find a partition of L into countable dense sets L_α for $\alpha < \omega_1$ so that each L_α is closed under f. By slicewise coverability it follows that the graph of f can be covered by countably many increasing functions, thus one of these has uncountable intersection with f. □

Homeomorphism Groups

BA can be reframed topologically as a statement about homeomorphism groups as follows:

For each $A, B \subseteq \mathbb{R}$ which are $\aleph_1$-dense there is an autohomeomorphism $h : \mathbb{R} \to \mathbb{R}$ so that $h``A = B$.

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For a general topological space X we can define a subset $A \subseteq X$ to be $\aleph_1$-dense if it has intersection size $\aleph_1$ with every nonempty open subset of X. In this context we can formulate a BA-type axiom for non linearly ordered spaces, denote BA(X):

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The difficulty is in lifting the ambient structure, as the following theorem shows.

Theorem (S.)

There is a perfect Polish space all of whose $\aleph_1$*-dense subsets are homeomorphic if and only if all perfect Polish spaces have all their* $\aleph_1$*-dense subsets homeomorphic if and only if there is a unique separable, metrizable, zero dimensional* $\aleph_1$*-crowded space.*

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Question

Does BA $(:= \mathrm{BA}(\mathbb{R}))$ *imply* $\mathrm{BA}(\mathbb{R}^n)$ *for any finite* $n > 1$*?*

THANK YOU!

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