A Preservation Theorem for Valued Structures

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Preservation theorem for relational structures

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DEF.:

A map $f : A^n \to A$ preserves $R \subseteq A^k$ if $\forall t^1, \dots, t^n \in R$, $f(t^1, \dots, t^n) \in R$. f is a polymorphism of $\mathbb{A}$ if it preserves all relations of $\mathbb{A}$.

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THEOREM (Geiger '68, Bodnarčuk et al. '69)

Let $\mathbb{A}$ have a finite domain A. Then $R \subseteq A^k$ is pp-definable in $\mathbb{A}$ iff it is preserved by all polymorphisms of $\mathbb{A}$.

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THEOREM (Bodirsky, Nešetřil '03)

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$\hookrightarrow$ preserration theorem $\Rightarrow$ polymorphisms determine complexity!

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valued structure Γ consists of:

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DEF.: VCSP(Γ)

Input: $u \in \mathbb{Q}$, an expression

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Observation: CSPs are VCSPs with costs 0 and ∞.

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$$R \subseteq A^k \rightsquigarrow (R)_a^b(\bar{x}) = \begin{cases} a & \bar{x} \in R \\ b & \bar{x} \notin R \end{cases}$$

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THEOREM (Kozik, Ochremiak '15; Kolmogorov, Krokhin, Rolínek '15; Bulatov '17; Zhuk '17)

For every finite-domain Γ, $\mathrm{VCSP}(\Gamma)$ is in P or NP-complete.

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$\langle \Gamma \rangle :=$ smallest set containing $\perp$, $(=)_0^\infty$, valued relations of Γ and is closed under sums of atomic expressions and all operators above

$\hookrightarrow$ valued relations expressible in Γ

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LEMMA (Bodirsky, S., Lutz '24)

If $\mathrm{Aut}(\Gamma)$ is oligomorphic and all valued relations of Δ are expressible in Γ, then $\mathrm{VCSP}(\Delta)$ reduces to $\mathrm{VCSP}(\Gamma)$ in polynomial time.

Fractional polymorphisms

DEF.: Γ valued structure over τ.
A probability distribution ω on operations $C^n \to C$ is a fractional polymorphism of Γ of arity n if for every $R \in \tau$ of arity k and every $t^1, \dots, t^n \in C^k$

$$E_\omega[f \mapsto R(f(t^1, \dots, t^n))] \leq \frac{1}{n} \sum_{j=1}^{n} R(t^j)$$ (ω improves R).

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PROPOSITION (Bodirsky, S., Lutz '24)

If $\mathrm{Aut}(\Gamma)$ is oligomorphic, then $\langle \Gamma \rangle \subseteq \mathrm{Imp}(\mathrm{fPol}(\Gamma))$.

VCSP preservation theorem

THEOREM (Fulla, Živný '16)

Let Γ have a finite domain C. Then $R: C^k \to \mathbb{Q} \cup \{\infty\}$ is expressible in Γ iff R is improved by all fractional polymorphisms of Γ. In symbols, $\mathrm{Imp}(\mathrm{fPol}(\Gamma)) = \langle \Gamma \rangle$.

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Conjecture: If $\mathrm{Aut}(\Gamma)$ is oligomorphic, then

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ISSUE: not improving R is not a closed condition in $\mathbb{Q}$!

So far so good

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↳ $\mathrm{fPol}^+(\Gamma) := \overline{\bigcup_{\omega \in \mathrm{fPol}(\Gamma)} \mathrm{Support}(\omega)}$ (in the topology of pointwise convergence)

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Problem: does not have complexity consequences

Dream: $\langle \Gamma \rangle$ is closed

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THEOREM

If two valued cores are fractionally homomorphically equivalent, then they are isomorphic.

If for every $R \subseteq C^k$, $(R)_0^\infty \in \langle \Gamma \rangle$ iff $(R)_0^\infty \in \mathrm{Imp}(\mathrm{fPol}(\Gamma))$, then Γ is fractionally homomorphically equivalent to a valued core Δ.

preserves complexity

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