(Co)limits of partial acts

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Partial S-acts

Definition

Let S be a monoid and A be a set. A partial mapping

$$\cdot : A \times S \rightharpoonup A, \quad (a, s) \mapsto a \cdot s,$$

is called a **partial right S-action** if for every $a \in A$ and $s, s' \in S$ the following conditions hold:

1. $\exists\, a \cdot s \;\&\; \exists\, (a \cdot s) \cdot s' \quad \Rightarrow \quad \exists\, a \cdot (ss') \;\&\; (a \cdot s) \cdot s' = a \cdot (ss')$;
2. $\exists\, a \cdot 1 \;\&\; a \cdot 1 = a$.

The pair $(A, \cdot)$ is called a **partial right S-act** and is denoted by A_S.

Morphisms

Definition

A mapping $f : A_S \to B_S$ is called

- a **homomorphism** of partial right S-acts, if for all $a \in A$ and $s \in S$ we have
$$\exists\, as \quad \Rightarrow \quad \exists\, f(a)s \ \&\ f(a)s = f(as);$$
- an **op-homomorphism** of partial right S-acts, if for all $a \in A$ and $s \in S$ we have
$$\exists\, f(a)s \quad \Rightarrow \quad \exists\, as \ \&\ f(as) = f(a)s;$$
- a **strong homomorphism** of partial right S-acts, if it is a homomorphism and an op-homomorphism.

Categories

notation	**objects**	**morphisms**
PAct_S	partial right S-acts	homomorphisms
$\mathsf{PAct}_S^{\mathsf{o}}$	partial right S-acts	op-homomorphisms
$\mathsf{PAct}_S^{\mathsf{s}}$	partial right S-acts	strong homomorphisms

Which limits and colimits are interesting?

category	products	equalizers	coproducts	coequalizers
PAct_S	not really	not really	not really	
PAct_S^o		not really	not really	not really
PAct_S^s		not really	not really	not really

Coequalizers in PAct_S

Proposition

The coequalizer of the homomorphisms $f : A_S \to B_S$ and $g : A_S \to B_S$ is the partial act $(B/\hat{\rho})_S$ with the quotient map $\pi_{\hat{\rho}} : B_S \to (B/\hat{\rho})_S$, where $\hat{\rho}$ is the congruence generated by a relation

$$\rho := \{(f(a), g(a)) \mid a \in A\} \subseteq B \times B.$$

Problem: How to define a congruence such that the quotient set would be a partial act, quotient map would be a homomorphism, and all the usual conditions of congruences would be satisfied?

Solution: Jerez, E. (2025). The category of partial group actions: quotients, (co)limits and groupoids. Communications in Algebra, 53(6), 2528–2554.

Definition (ρ-chain)

Let us have an equivalence relation $\rho \subseteq A \times A$ and

- $a'_0, a'_1, \dots, a'_{n-1} \in A$,
- $s_1, s_2, \dots, s_n \in S$,

such that the products $a'_{i-1}s_i =: a_i$ exists and $a_i \rho a'_{i+1}$. Then the following sequence is called a ρ-chain from a_0 to a'_n:

$$
\begin{array}{ccccccccccc}
a_0 & & a'_1 & \xrightarrow{s_2} & a_2 & & & & a'_{n-1} & \xrightarrow{s_n} & a_n \\
\rho & & \rho & & \rho & & & \iddots & & & \rho \\
a'_0 & \xrightarrow{s_1} & a_1 & & a'_2 & \xrightarrow{s_3} & a_3 & & & & a'_n
\end{array}
$$

Congruences in PAct_S

Definition (Congruence)

For every two ρ-chains, if $a_0 \rho b_0$ and $s_1 s_2 \dots s_n = t_1 t_2 \dots t_k$, then $a'_n \rho b'_k$.

$$
\begin{array}{ccccccccccccc}
a'_0 & \xrightarrow{s_1} & a_1 & & a'_2 & \xrightarrow{s_3} & a_3 & & & & & & \\
\rho & & \rho & & \rho & & & \ddots & & & & & \\
a_0 & & a'_1 & \xrightarrow{s_2} & a_2 & & & & a_{n-1} & \xrightarrow{s_n} & a_n & \rho & a'_n \\
\rho & & & & & & & & & & & & \rho \\
b_0 & & b'_1 & \xrightarrow{t_2} & b_2 & & & & b'_{k-1} & \xrightarrow{t_k} & b_k & \rho & b'_k \\
\rho & & \rho & & \rho & & & \iddots & & & & & \\
b'_0 & \xrightarrow{t_1} & b_1 & & b'_2 & \xrightarrow{t_3} & b_3 & & & & & &
\end{array}
$$

Quotient partial acts in PAct_S

Proposition

We can view the quotient set A/ρ as a partial right S-act if we define

$$\exists [a]s \overset{def}{\Leftrightarrow} \textit{there exists a } \rho\textit{-chain such that } a_0 = a \textit{ and } s_1 s_2 \ldots s_n = s,$$

in which case we set

$$[a]s := [a'_n].$$

category	products	equalizers	coproducts	coequalizers
PAct_S	not really	not really	not really	✓
$\mathrm{PAct}^{\mathrm{o}}_S$		not really	not really	not really
$\mathrm{PAct}^{\mathrm{s}}_S$		not really	not really	not really

Products in PAct_S°

Problem: The Cartesian product $A \times B$ with the naive partial action

$$\exists\,(a,b)s := (as, bs) \quad \Leftrightarrow \quad \exists\, as \;\; \& \;\; \exists\, bs$$

does not allow the coordinate projections $p_A : A \times B \to A$, $p_B : A \times B \to B$ to be op-homomorphisms.

Solution: Consider a set

$$P(A,B) := \{\varphi : S \rightharpoonup A \times B \mid \varphi \text{ satisfies some specific conditions } \}.$$

Partial S-action on the set $P(A,B)$ is defined as

$$\exists\, \varphi s \quad \Leftrightarrow \quad s \in \mathrm{dom}(\varphi).$$

If the product φs exists, we define φs to be the partial map $S \rightharpoonup A \times B$ with

$$(\varphi s)(s') = \varphi(ss') \quad \text{whenever} \quad s' \in \mathrm{dom}(\varphi s) := \{s' \in S \mid ss' \in \mathrm{dom}(\varphi)\}.$$

Products in PAct_S°

Proposition

The product of partial S-acts A_S and B_S is the partial S-act

$$A_S \prod B_S = \{\varphi^{\diamond} \mid \varphi \in P(A,B)\}$$

with op-homomorphisms

$$p_A \circ \mathrm{ev}_1 : A_S \prod B_S \to A_S,$$
$$p_B \circ \mathrm{ev}_1 : A_S \prod B_S \to B_S,$$

where $\varphi^{\diamond}$ is a restriction of φ with the minimal domain with respect to the requirements of the elements of $P(A,B)$ and $ev_1 : \varphi^{\diamond} \mapsto \varphi^{\diamond}(1)$.

category	products	equalizers	coproducts	coequalizers
PAct_S	not really	not really	not really	✓
PAct_S^{o}	✓	not really	not really	not really
PAct_S^{s}		not really	not really	not really

Terminal object in PAct^{s}

Problem: The terminal object as the product of a empty family of partial S-acts, has to have more than one element, in general.
Solution: The terminal object in PAct^{s}_{S} is the partial act

$$T_S = \{\varphi : S \rightharpoonup \wp(S) \mid \varphi \text{ satisifies some conditions } \}.$$

The unique strong homomorphism

$$\Phi_A : A_S \to T_S, \quad a \mapsto \varphi_a,$$

is defined as follows:

$$\mathrm{dom}(\varphi_a) = \{s \in S \mid \exists\, as\} \quad \text{and} \quad \varphi_a(s) = \{s' \in S \mid \exists\, (as)s'\}.$$

Products in PAct_S^s

Proposition

The product of partial right S-acts A_S and B_S is the partial S-act

$$A_S \prod B_S = \{(a,b) \mid \exists \varphi \in T : \Phi_A(a) = \varphi = \Phi_B(b)\} \subseteq A \times B$$

with the usual coordinate projections.

category	products	equalizers	coproducts	coequalizers
PAct_S	not really	not really	not really	✓
PAct_S^{o}	✓	not really	not really	not really
PAct_S^{s}	✓	not really	not really	not really

Thank you for listening!