2-generated minimal Taylor algebras on a 4-element set

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February, 2026

Motivation

The colored edge theory invented by Andrei Bulatov in his proof of the Dichotomy Theorem defines edges in an algebra based on its 2-generated subalgebras. If every 2-generated minimal Taylor algebra with at least three elements were either affine or had a nontrivial congruence, then Bulatov's theory could be significantly simplified.

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Brady's Notes on CSPs and Polymorphisms (Problem 4.2.2)

Is there any minimal Taylor algebra which is simple, is generated by two elements, has size at least 3, and is not affine?

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Is there any minimal Taylor algebra which is simple, is generated by two elements, has size at least 3, and is not affine?

[BBBKZ] Conjecture 5.17

If $\mathbf{A}$ is a minimal Taylor algebra which is generated by two elements $a, b \in A$ such that neither (a, b) nor (b, a) is an edge, then there are proper 3-absorbing subuniverses $C, D \trianglelefteq_3 \mathbf{A}$ such that $a \in C$ and $b \in D$.

Minimal Taylor algebras (MTA)

Definition

A finite algebra $\mathbf{A}$ is a minimal Taylor algebra if it is Taylor, and it has no proper reduct that is also Taylor.

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Proposition (BBBKZ)

Any subalgebra, finite power, or quotient of a MTA is a MTA.

Proposition (BBBKZ)

Let $\mathbf{A}$ *be a MTA and* $B \subseteq A$ *be closed under an operation* $f \in \mathrm{Clo}(\mathbf{A})$ *such that* B *together with the restriction of* f *to* B *forms a Taylor algebra. Then* B *is a subuniverse of* $\mathbf{A}$.

Edges

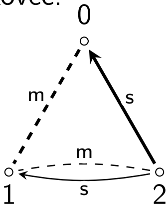
Definition (BBBKZ)

Let **A** be an algebra and $a, b \in A$.

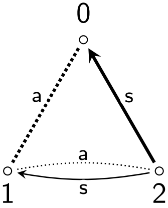
- (a, b) is a weak semilattice edge if there is a proper congruence θ on $\mathrm{Sg}\{a, b\}$ and a binary term t such that $t(a/\theta, b/\theta) = t(b/\theta, a/\theta) = b/\theta$.
- $\{a, b\}$ is a weak majority edge if there is a proper congruence θ on $\mathrm{Sg}\{a, b\}$ and a term $m \in \mathrm{Clo}_3(\mathbf{A})$ which acts as the majority operation on $\{a/\theta, b/\theta\}$.
- $\{a, b\}$ is a weak affine edge if there is a proper congruence θ on $\mathrm{Sg}\{a, b\}$ and a term operation $p \in \mathrm{Clo}_3(\mathbf{A})$ such that $(\mathrm{Sg}\{a, b\}/\theta; p)$ is an affine Mal'cev algebra with respect to some abelian group $(\mathrm{Sg}\{a, b\}/\theta; +)$.

An edge (a, b) is called strong if for some maximal congruence θ witnessing the edge and every $a', b' \in A$ such that $(a, a'), (b, b') \in \theta$, we have $\mathrm{Sg}\{a, b\} = \mathrm{Sg}\{a', b'\}$.

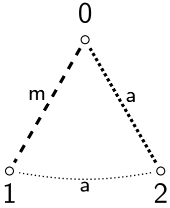
Post lattice: there are only three MTA on a two-element domain - the semilattice, majority algebra and affine Mal'cev algebra.
Brady's Notes: there are 24 MTA on a three-element domain, among which 5 are 2-generated. You can also find them in the master's thesis of Filip Jankovec.



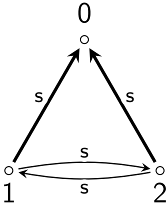
$\mathbf{T_1^N}$



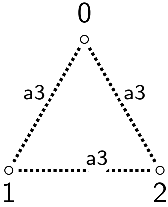
$\mathbf{T_2^N}$



$\mathbf{T_3^N}$



$\mathbf{T_4^N}$



$\mathbf{T_5^N} \cong \mathbb{Z}_3^{\mathbf{aff}}$

Absorption

Definition (BBBKZ)

Let $\mathbf{A}$ be an algebra and $B \subseteq A$. We call B an *n-absorbing set* of $\mathbf{A}$ if there is a term operation $t \in \mathrm{Clo}_n(\mathbf{A})$ such that $t(\mathbf{a}) \in B$ whenever $\mathbf{a} \in A^n$ and $|\{i : a_i \in B\}| \geqslant n-1$. If, additionally, B is a subuniverse of $\mathbf{A}$, we write $B \trianglelefteq_n \mathbf{A}$ (B n-absorbs $\mathbf{A}$ by t).

Theorem (BBBKZ)

Let $\mathbf{A}$ be a MTA and B an n-absorbing set of $\mathbf{A}$. Then B is a subuniverse of $\mathbf{A}$.

Proposition (BBBKZ)

Let $\mathbf{A}$ be a MTA and $B, C \subseteq A$. The following hold:

(1) *If $B, C \trianglelefteq_3 \mathbf{A}$, then $B \cup C \leq \mathbf{A}$ and $B \cap C \trianglelefteq_3 \mathbf{A}$.*

(2) *If $C \trianglelefteq_3 B \trianglelefteq_3 \mathbf{A}$, then $C \trianglelefteq_3 \mathbf{A}$.*

Proposition (BBBKZ)

Let $\mathbf{A}$ *be a MTA and* $B, C \subseteq A$. *The following hold:*

(1) *If* $B, C \trianglelefteq_2 \mathbf{A}$, *then* $B \cup C \leq \mathbf{A}$ *and* $B \cap C \trianglelefteq_2 \mathbf{A}$.

(2) *If* $C \trianglelefteq_2 B \trianglelefteq_2 \mathbf{A}$, *then* $C \trianglelefteq_2 \mathbf{A}$.

Lemma (BBBKZ)

Let $\mathbf{A}$ *be a MTA and* $B \subseteq A$. *The following are equivalent.*

- $B \trianglelefteq_2 \mathbf{A}$.
- $\mathbf{B}$ *is strongly absorbing subalgebra of* $\mathbf{A}$, *i.e. for any term* $t(x_1, \ldots, x_n)$ *and any essential position* x_i *of* $t^{\mathbf{A}}$, *if* $a_1, \ldots, a_n \in A$ *and* $a_i \in B$, *then* $t^{\mathbf{A}}(a_1, \ldots, a_n) \in B$.

Theorem (BBBKZ)

If $\mathbf{A}$ *is a MTA that is generated by two distinct elements* $a, b \in A$*, then either* $\mathbf{A}$ *has a nontrivial abelian quotient, or at least one of* a, b *is contained in a proper ternary absorbing subuniverse of* $\mathbf{A}$*.*

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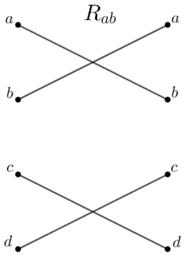
Definition

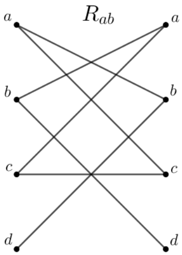
Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be algebras and $R \leq_{sd} \mathbf{A}_1 \times \mathbf{A}_2$. For any $i \in \{1, 2\}$, the i-th *link tolerance* of R, denoted by $tol_i R$ is defined by

$$tol_1 R := \{(a_1, a_1') \in A_1^2 : (\exists a_2 \in A_2)\ (a_1, a_2) \in R \text{ and } (a_1', a_2) \in R\},$$
$$tol_2 R := \{(a_2, a_2') \in A_2^2 : (\exists a_1 \in A_1)\ (a_1, a_2) \in R \text{ and } (a_1, a_2') \in R\}.$$

The transitive closure of $tol_i R$ is the i-th *link congruence* of R, denoted by $lk_i R$. We say R is *linked* if its link congruences are full.

When a MTA $\mathbf{A} = \mathrm{Sg}\{a, b\}$ is simple, it is useful to consider the algebra $R_{ab} = \mathrm{Sg}\{(a, b), (b, a)\} \leq_{sd} \mathbf{A}^2$. Then R_{ab} is either the graph of automorphism φ of $\mathbf{A}$ such that $\varphi(a) = b$ and $\varphi(b) = a$, or the link congruence $lk_1 R_{ab}$ is not the identity, which, since $\mathbf{A}$ is simple, means that R_{ab} is linked.



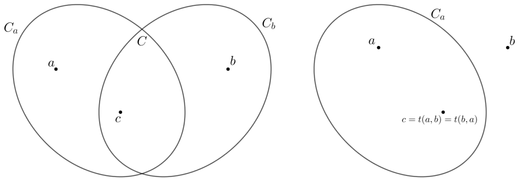


Theorem (Barto, Kozik: *Loop Lemma*)

Let $\mathbf{A}$ be a finite Taylor algebra and $R \leq_{sd} \mathbf{A}^2$ is linked. Then R contains a loop, that is $(c, c) \in R$ for some $c \in A$.

Lemma

Let **A** *be a MTA of size at least* 3 *generated by two distinct elements* $a, b \in A$. *Let* $C \trianglelefteq_3 \mathbf{A}$ *and let* $a, c \in C$ *such that* $c = t(a, b) = t(b, a)$ *for some binary term* t. *If* $\{a, c\}$ *and* $\{b, c\}$ *are subuniverses, then* $\mathrm{Sg}\{a, b\} = \{a, b, c\}$.



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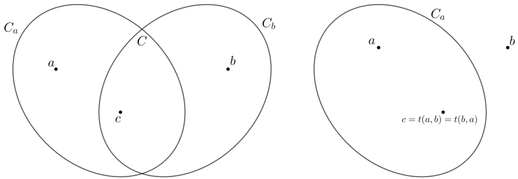


Theorem (Barto, Kozik)

A *is Taylor iff for every prime* $p > |A|$, **A** *has an idempotent term operation* g *of arity* p *which is cyclic, that is, for any* $\mathbf{x} \in A^p$,

$$g(x_1, x_2, \dots, x_p) = g(x_2, \dots, x_p, x_1).$$

Lemma

Let $\mathbf{A}$ *be a simple nonabelian MTA of size at least* 3 *generated by two distinct elements* $a, b \in A$ *and* $a \in C_a \trianglelefteq_3 \mathbf{A}$.

a) *If* R_{ab} *is the graph of automorphism* φ *which swaps* a *and* b, *and* g *is a cyclic term of algebra* $\mathbf{A}$ *of arity* n, *then there exists a cyclic term* g' *of the same arity such that*

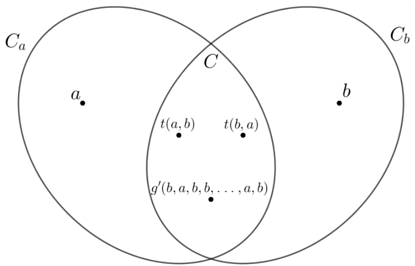
$$g'(\{a, b\}^n \setminus \{(a, a, \ldots, a), (b, b. \ldots, b)\}) \subseteq C_a \cap C_b,$$

where C_b *is the image of* C_a *under* φ, *thus* $b \in C_b \trianglelefteq_3 \mathbf{A}$.

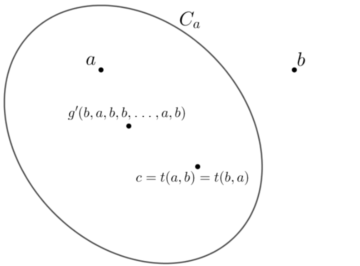
b) *If* R_{ab} *is linked and* g *is a cyclic term of algebra* $\mathbf{A}$ *of arity* n, *then there exists a cyclic term* g' *of the same arity such that*

$$g'(\{a, b\}^n \setminus \{(b, b, \ldots, b)\}) \subseteq C_a.$$

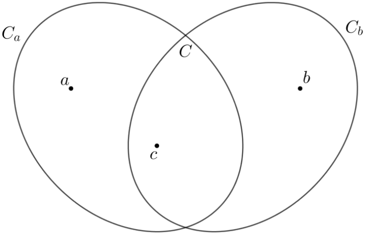
a)

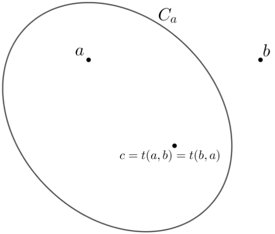


b)



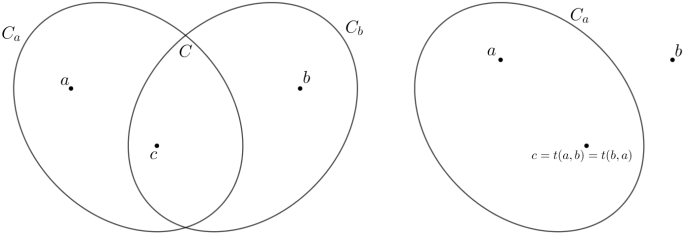
The general case





$\{a\} \ntrianglelefteq_3 \mathbf{A}$ and $\{b\} \ntrianglelefteq_3 \mathbf{A}$.
Automorphism case: $C \neq \emptyset$ and for $|A| = 4$ must be $C = \{c, d\}$.
Linked case: $\{a, c\}$ and $\{b, c\}$ cannot both be subuniverses.

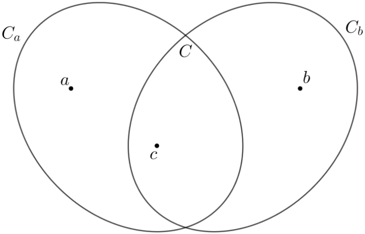
The general case

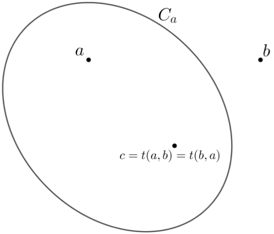


Theorem (BBBKZ)

If $\mathbf{A}$ *is a MTA and* $a \in A$ *satisfies that there is no outgoing weak semilattice edge, nor weak affine edge, connecting* a *and any other element, then* $\{a\} \trianglelefteq_3 \mathbf{A}$.

The general case





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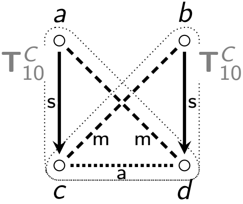
If $\mathbf{A}$ *is a MTA and* $a \in A$ *satisfies that there is no outgoing weak semilattice edge, nor weak affine edge, connecting a and any other element, then* $\{a\} \trianglelefteq_3 \mathbf{A}$.

Theorem (Bulatov's Rectangularity Theorem)

Let $\mathbf{R} \leq_{sd} \mathbf{A}^2$ *and* $\mathbf{A}$ *is a MTA. If* B *is sink strong* $a_s s_s$*-component of* $\mathbf{A}$, R *is linked and* $R \cap B^2 \neq \emptyset$, *then* $B^2 \subseteq R$.

Some subcase of the automorphism case



We can find a ternary cyclic operation g such that $g(a, a, b) = d$ and $g(b, b, a) = c$.

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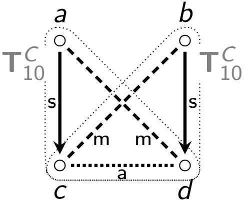

We can find a ternary cyclic operation g such that $g(a, a, b) = d$ and $g(b, b, a) = c$.Let g' be a ternary cyclic term defined by

$$g'(x, y, z) := g(g(x, x, g(x, y, z)), g(y, y, g(y, z, x)), g(z, z, g(z, x, y))).$$

We have $g'(a, a, b) = g(g(a, a, d), g(a, a, d), g(b, b, d)) = g(a, a, d) = a$ and $g'(b, b, a) = b$. Since g' is cyclic, we get $\mathrm{Sg}\{a, b\} = \{a, b\}$, a contradiction.

The main theorem

Theorem

Let $\mathbf{A}$ *be a nonabelian MTA on a domain of size four generated by two distinct elements* $a, b \in A$. *Then* $\mathbf{A}$ *is not simple.*

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Let **A** *be a nonabelian MTA on a domain of size four generated by two distinct elements* $a, b \in A$*. Then* **A** *is not simple.*

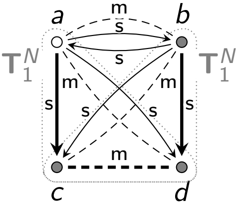
Theorem

Any 2-generated MTA on a domain of size four is not simple.

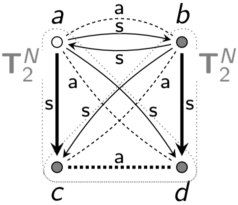
Proof. If **A** is a 2-generated abelian MTA on a four-element domain, then it has to be term-equivalent to an affine Mal'cev algebra $(A; x - y + z)$ over $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. $\square$

Classification of 2-generated MTA on a domain of size 4

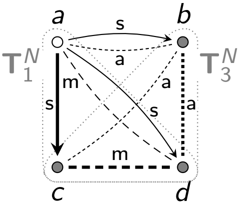
We proved that every 2-generated MTA on a four-element domain is not simple. We classified these algebras based on their maximal quotients. Each MTA $\mathbf{A} = \mathrm{Sg}\{a, b\}$ on a four-element set must have at least one of the following quotients: two-element semilattice, two-element majority algebra, or one of the affine algebras $\mathbb{Z}_2^{\mathsf{aff}}$ and $\mathbb{Z}_3^{\mathsf{aff}}$.



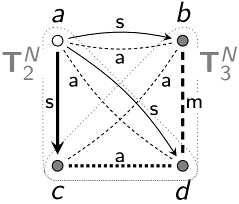
$\mathbf{T}_{4,1}$



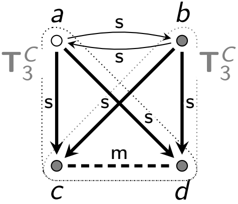
$\mathbf{T}_{4,2}$



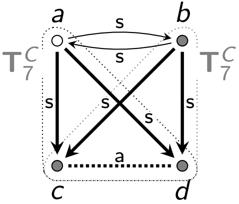
$\mathbf{T}_{4,3}$



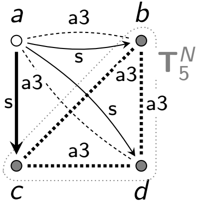
$\mathbf{T}_{4,4}$



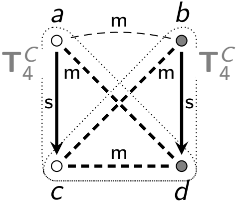
$\mathbf{T}_{4,5}$



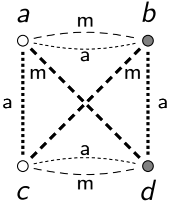
$\mathbf{T}_{4,6}$



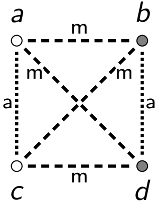
$\mathbf{T}_{4,7}$



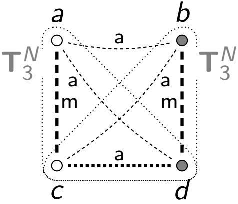
$\mathbf{T}_{4,8}$



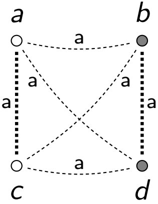
$\mathbf{T}_{4,9}$



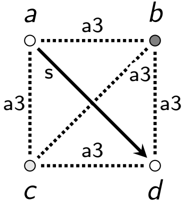
$\mathbf{T}_{4,10}$



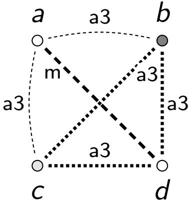
$\mathbf{T}_{4,11}$



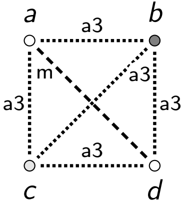
$\mathbf{T}_{4,12}$, $\mathbf{T}_{4,13}$



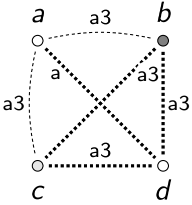
$\mathbf{T}_{4,14}$



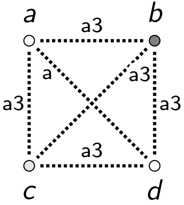
$\mathbf{T}_{4,15}$



$\mathbf{T}_{4,16}$



$\mathbf{T}_{4,17}$



$\mathbf{T}_{4,18}$

Thank you for your attention!