Boolean $\pm$-preclones

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In memoriam Günther Eigenthaler

This is the first AAA attended by me since the passing of Günther Eigenthaler. Therefore I dedicate this talk to the memory of my esteemed colleague and dear friend Günther.

GÜNTHER EIGENTHALER
9.2.1950 - 14.2.2025

Günther, Laci Márki, R.P.
September 2023 during an excursion (Lunzer See) in Austria

Outline

±-preclones (S-preclones)

±-relations, ±-preservation and a Galois connection

The lattice ${}^{\pm}\mathcal{L}_A$ of Boolean ±-preclones

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$\pm$-preclones (S-preclones)

$\pm$-relations, $\pm$-preservation and a Galois connection

The lattice ${}^{\pm}\mathcal{L}_A$ of Boolean $\pm$-preclones

Motivating example

some history:
Nov. 2021 PALS talk by P. Jipsen on partially ordered algebras (and po-clones): operations which in each argument are *order-preserving* **or** *order-reversing* (for some given order on the base set).

Questions: how to characterize such "po-clones"?

R.P.: characterization via invariant relations?

Analogies to many-sorted algebras
(results of E. Lehtonen/ R. Pöschel/ T. Waldhauser),

Let P be a property for unary functions $g \in A^A$.
"motivating example": $P = +$: order-preserving
$P = -$: order-reversing

An n-ary *operation* $f(x_1, \ldots, x_n)$ *has property* P *in an argument*, say x_1, $:\Longleftrightarrow$ each translation $x_1 \mapsto f(x_1, c_2, \ldots, c_n)$ has this property P (for all constants $c_2, \ldots, c_n \in A$).

How to handle composition? order-reversing composed with order-reversing is order-preserving! Formalization: Collect the properties in a monoid $S = (\{+, -\}, \cdot)$, here a group

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±-operations (S-operations)

$S := \pm := \{+, -\}$ (in general, finite monoid S).

n-ary *S-operation ($\pm$-operation)*: operation f together with its *signum*

$$f : A^n \to A \text{ with } \operatorname{sgn}(f) = (s_1, \dots, s_n) \in S^n,$$

i.e., the i-th argument of f gets a label (*sign*) $s_i \in S$ ($i = 1, \dots, n$).

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(Boolean) ±-preclones

±-preclone := set $F \subseteq {}^{\pm}\mathrm{Op}(A)$ of ±-operations closed under:

(1) $\mathrm{id}_A \in F$, $\mathrm{id}_A(x) = x$, $\mathrm{sgn}(\mathrm{id}_A) := (+)$ (+ unit element of S),

(2) permutation of arguments (operations ζ, τ),

(3) identification of arguments *with the same sign s* (Δ^s),

(4) adding fictitious arguments of (arbitrary) sign $s \in S$,
e.g., $(\nabla^s f)(x_1, x_2, \ldots, x_{n+1}) := f(x_2, \ldots, x_{n+1})$, where
$\mathrm{sgn}(\nabla^s f) = (s, s_1, \ldots, s_n)$ for $\mathrm{sgn}(f) = (s_1, \ldots, s_n)$,

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$\mathrm{sgn}(f) = (s_1, \ldots, s_n)$ and $\mathrm{sgn}(g) = (s'_1, \ldots, s'_m)$. Then

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$\mathrm{sgn}(f) = (s_1, \dots, s_n)$ and $\mathrm{sgn}(g) = (s'_1, \dots, s'_m)$. Then

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$^{\pm}\langle F \rangle$:= ±-preclone generated by $F \subseteq {}^{\pm}\mathrm{Op}(A)$.

Boolean ±-preclone if $A = \{0, 1\}$ [Remark: preclone = operad]

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$\pm$-preclone := set $F \subseteq {}^{\pm}\mathrm{Op}(A)$ of $\pm$-operations closed under:

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Example (composition)

properties $S = \{+, -\}$, $A = \{0, 1\}$ with order $0 < 1$,
$+$ means order preserving, $-$ means order reversing
(such functions really form a ±-preclone).

Composition:
$f(x_1, x_2) = \neg x_1 \wedge x_2$, $\operatorname{sgn}(f) = (s_1, s_2) = (-, +)$,
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$(f \circ g)(x_1, x_2, x_3) = f(g(x_1, x_2), x_3) = \neg(x_1 \vee \neg x_2) \wedge x_3 = \neg x_1 \wedge x_2 \wedge x_3$
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One is allowed to identify x_2 and x_3, but not x_2 and x_1.

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Outline

$\pm$-preclones (S-preclones)

$\pm$-relations, $\pm$-preservation and a Galois connection

The lattice ${}^{\pm}\mathcal{L}_A$ of Boolean $\pm$-preclones

±-relations (S-relations) and ±-preservation $\overset{\pm}{\triangleright}$

recall: $S = \pm := \{+, -\}$

m-ary *±-relation*: $\varrho = (\varrho_+, \varrho_-)$ with $\varrho_s \subseteq A^m$ $(s \in S)$

classical notion of preservation: $f \triangleright \varrho :\iff f(\varrho, \dots, \varrho) \subseteq \varrho$

The "S-version":

$f \in {}^{\pm}\mathrm{Op}^{(n)}(A)$, $\mathrm{sgn}(f) = (s_1, \dots, s_n)$, $\varrho = (\varrho_+, \varrho_-) \in {}^{\pm}\mathrm{Rel}^{(m)}(A)$

$$f \overset{\pm}{\triangleright} (\varrho_+, \varrho_-) :\iff f(\varrho_{s_1}, \dots, \varrho_{s_n}) \subseteq \varrho_+ \text{ and } f(\varrho_{s_1 \cdot -}, \dots, \varrho_{s_n \cdot -}) \subseteq \varrho_-$$

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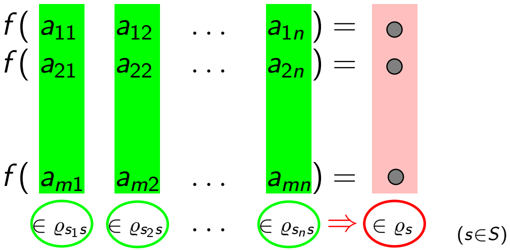

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$$
\begin{array}{lcccccl}
f(& a_{11} & a_{12} & \dots & a_{1n} &) = & \bullet \\
f(& a_{21} & a_{22} & \dots & a_{2n} &) = & \bullet \\
 & \vdots & & & & & \\
f(& a_{m1} & a_{m2} & \dots & a_{mn} &) = & \bullet \\
 & \in \varrho_{s_1 s} & \in \varrho_{s_2 s} & \dots & \in \varrho_{s_n s} & \Rightarrow & \in \varrho_s
\end{array}
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The Galois connection ${}^{\pm}\mathrm{Pol} - {}^{\pm}\mathrm{Inv}$

$\overset{\pm}{\triangleright}$ induces a Galois connection with the operators

$${}^{\pm}\mathrm{Pol}\, Q := \{f \in {}^{\pm}\mathrm{Op}(A) \mid \forall \varrho \in Q\colon f \overset{\pm}{\triangleright} \varrho\} \quad (\pm\text{-polymorphisms}),$$

$${}^{\pm}\mathrm{Inv}\, F := \{\varrho \in {}^{\pm}\mathrm{Rel}(A) \mid \forall f \in F\colon f \overset{\pm}{\triangleright} \varrho\} \quad (\text{invariant } \pm\text{-relations}).$$

for $F \subseteq {}^{\pm}\mathrm{Op}(A)$ and $Q \subseteq {}^{\pm}\mathrm{Rel}(A)$.

Theorem (The Galois closures)

$${}^{\pm}\langle F\rangle = {}^{\pm}\mathrm{Pol}\,{}^{\pm}\mathrm{Inv}\, F \qquad (\pm\text{-preclone generated by } F),$$

$${}^{\pm}[Q] = {}^{\pm}\mathrm{Inv}\,{}^{\pm}\mathrm{Pol}\, Q \qquad (\pm\text{-relational clone generated by } Q).$$

[JipLP2023]: *S-preclones and the Galois connection ${}^{S}\mathrm{Pol} - {}^{S}\mathrm{Inv}$, Part I, Algebra Universalis 85, 2024*

The Galois connection ${}^{\pm}\mathrm{Pol} - {}^{\pm}\mathrm{Inv}$

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once more: our "motivating" Example

$(A, \leq)$ poset $(0 < 1)$, $S = \{+, -\}$ (group).

For the ±-preclone F (of ±-operations where $+$ means order preserving und $-$ means order reversing) we have the following relational characterization:

$$F = {}^{\pm}\mathrm{Pol}\,\varrho \text{ for the } \pm\text{-relation } \varrho = (\varrho_+, \varrho_-) := (\leq, \geq).$$

Example: For $g(x_1, x_2) = x_1 \vee \neg x_2$, $\mathrm{sgn}(g) = (+, -)$,
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$g\left(\begin{smallmatrix}0\\1\end{smallmatrix}, \begin{smallmatrix}c\\c\end{smallmatrix}\right) \in f(\varrho_+, \varrho_-) \subseteq \varrho_+ = \left(\begin{smallmatrix}0&0&1\\0&1&1\end{smallmatrix}\right)$, i.e., order-preserving in x_1,

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$g\left(\begin{smallmatrix}c\\c\end{smallmatrix}, \begin{smallmatrix}0\\1\end{smallmatrix}\right) \in f(\varrho_-, \varrho_+) = f(\varrho_{+\cdot -}, \varrho_{-\cdot -}) \subseteq \varrho_- = \left(\begin{smallmatrix}0&1&1\\0&0&1\end{smallmatrix}\right)$, i.e., order-reversing in x_2.

once more: our "motivating" Example

$(A, \leq)$ poset $(0 < 1)$, $S = \{+, -\}$ (group).

For the ±-preclone F (of ±-operations where $+$ means order preserving und $-$ means order reversing) we have the following relational characterization:

$$F = {}^{\pm}\mathrm{Pol}\,\varrho \text{ for the } \pm\text{-relation } \varrho = (\varrho_+, \varrho_-) := (\leq, \geq).$$

Example: For $g(x_1, x_2) = x_1 \vee \neg x_2$, $\mathrm{sgn}(g) = (+, -)$,
$g \overset{\pm}{\triangleright} \varrho$ implies

$g\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}, \begin{smallmatrix} c \\ c \end{smallmatrix}\right) \in f(\varrho_+, \varrho_-) \subseteq \varrho_+ = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{smallmatrix}\right)$, i.e., order-preserving in x_1,

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Outline

$\pm$-preclones (S-preclones)

$\pm$-relations, $\pm$-preservation and a Galois connection

The lattice ${}^{\pm}\mathcal{L}_A$ of Boolean $\pm$-preclones

Some properties of the lattice $^{\pm}\mathcal{L}_A$

$^{\pm}\mathcal{L}_A$:= lattice of all ±-preclones on A w.r.t. $\subseteq$
($^{\pm}\mathcal{L}_2$ for Boolean ±-preclones, $A = \{0, 1\}$)

Some properties (hold also for arbitrary monoids S instead of ±)

- least ±-preclone: $^{\pm}J_A$ = ±-projections = $^{\pm}\langle \mathrm{id}_A \rangle$
- largest ±-preclone: $^{\pm}\mathrm{Op}(A)$
- $^{\pm}\mathcal{L}_A$ is atomic and coatomic (each ±-preclone contains an atom and is contained in a coatom).
- $^{\pm}\mathcal{L}_A$ has finitely many atoms and coatoms.

$^{\pm}\mathrm{Op}(A)$ is finitely generated (by at most binary ±-operations)
$^{\pm}\mathrm{Rel}(A)$ is finitely generated (by at most ternary ±-relations)

Problem: Describe all maximal or minimal ±-preclones (coatoms or atoms)

Recall: $\mathcal{L}_2$, the Post lattice of Boolean clones, is countable and has 5 maximal and 7 minimal clones.

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Theorem

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(a) ${}^{\pm}\mathrm{Pol}(\sigma, \sigma)$ *with* $\sigma \in \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ *where* $\mathrm{Pol}\,\sigma_i$ *is maximal in* $\mathcal{L}_2$ *(0-preserving, 1-preserving, monotone, self-dual, linear operations)*
$\sigma_0 = \{0\}$, $\sigma_1 = \{1\}$, $\sigma_2 = \leq = \{(0,0),(0,1),(1,1)\}$,
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(b) ${}^{\pm}\mathrm{Pol}(\leq, \geq)$ *our motivating example!* *all $\pm$-operations where each $+$argument is order-preserving and each $-$argument is order-reversing.*

(c) ${}^{\pm}\mathrm{Pol}(A, \emptyset) =$ *all functions with positive or mixed signum.*

(d) ${}^{\pm}\mathrm{Pol}(A^2, \Delta_A) =$ *all Boolean $\pm$-operations, where each negative argument is fictitious (including all negative constants).*

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The minimal Boolean ±-preclones

Theorem

There are twenty minimal Boolean ±-preclones. Each such ±-preclone is of the form ${}^{\pm}\langle f\rangle$ with one ±-operation f as generator:

(A) ${}^{\pm}\langle (x\wedge y)\vee(y\wedge z)\vee(z\wedge x)\rangle, {}^{\pm}\langle x+y+z\rangle$
where the generators have signum $\lambda=(+,+,+,-)$,
(majority and minority operation, the last argument is ficticious) (#2)

(B) ${}^{\pm}\langle h_0\rangle, {}^{\pm}\langle h_1\rangle, {}^{\pm}\langle h_y\rangle$ *where* $h_\dagger(x,y,z,u)=\begin{cases} x & \text{if } x=y \text{ or } z=u,\\ \dagger & \text{otherwise,}\end{cases}$
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(C) ${}^{\pm}\langle x\wedge y\rangle, {}^{\pm}\langle x\vee y\rangle, {}^{\pm}\langle x\vee(y\wedge z)\rangle, {}^{\pm}\langle x\wedge(y\vee z)\rangle,$
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(D) $^{\pm}\langle 0\rangle, {}^{\pm}\langle 1\rangle, {}^{\pm}\langle y\rangle, {}^{\pm}\langle \neg y\rangle, {}^{\pm}\langle \neg x\rangle, {}^{\pm}\langle x\wedge y\rangle, {}^{\pm}\langle x\vee y\rangle$
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Preimage classes

(a tool for investigations of the structure of $^{\pm}\mathcal{L}_2$)

Let $F \in \mathcal{L}_2$ be a Boolean clone.

$$F^{\boxdot} := \{P \in {}^{\pm}\mathcal{L}_2 \mid \langle \mathring{P} \rangle = F\}$$ *preimage class of F*

$\langle \mathring{P} \rangle$ = "underlying clone" forgetting all signa

Remark: relational characterization $\langle \mathring{P} \rangle = \mathrm{Pol}\{\sigma \in \mathrm{Rel}(A) \mid (\sigma, \sigma) \in {}^{\pm}\mathrm{Inv}\, P\}$

Structure: semi-interval with greatest element $P_F := \{f \in {}^{\pm}\mathrm{Op}(A) \mid \mathring{f} \in F\}$

The lattice $^{\pm}\mathcal{L}_2$ of Boolean ±-preclones is the (disjoint) union of all preimage classes of Boolean clones:

$$^{\pm}\mathcal{L}_2 = \bigcup \{F^{\boxdot} \mid F \in \mathcal{L}_2\}.$$

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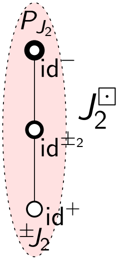
How preimage classes look like? Some examples

unary clones F (i.e., generated by unary Boolean functions):

$F = J_2 := \langle \mathrm{id} \rangle$: $|J_2^{\boxdot}| = 3$



$\mathrm{id}^-(x) = x$

$\mathrm{sgn}(\mathrm{id}^-) = (-)$

$\mathrm{id}^{\pm_2}(x, y) = y$

$\mathrm{sgn}(\mathrm{id}^{\pm_2}) = (+, -)$

$\mathrm{id}^+(x) = x$

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$F_0 := \langle c_0 \rangle$: $|F_0^{\boxdot}| = 6$,
$F_1 := \langle c_1 \rangle$: $|F_1^{\boxdot}| = 6$,
$F_{01} := \langle c_0, c_1 \rangle$: $|F_{01}^{\boxdot}| = 10$,
$F_\neg := \langle \neg \rangle$: $|F_\neg^{\boxdot}| = 7$,
$F_2 := \langle \mathrm{Op}^{(1)}(A) \rangle$: $|F_2^{\boxdot}| = 19$.

There are 12 join-irreducible elements generated by
$\mathrm{id}_A^{\pm_2}, \mathrm{id}_A^-, c_0^{\pm}, c_0^+, c_0^-, c_1^{\pm}, c_1^+, c_1^-, \neg^{\pm_1}, \neg^{\pm_2}, \neg^+, \neg^-$.

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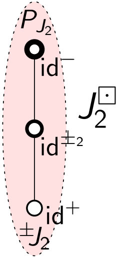

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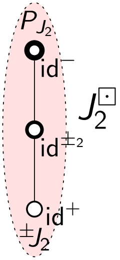

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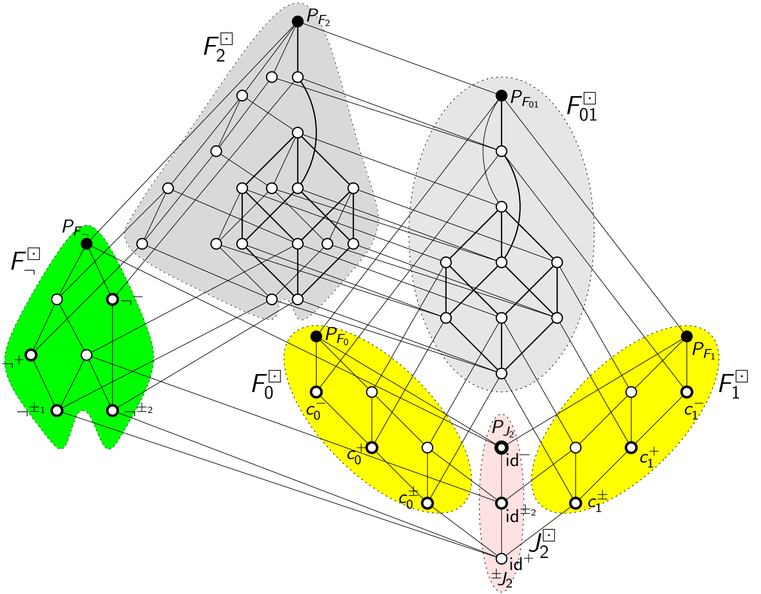
The lattice of unary ±-preclones



A challenging open problem

up to now, we investigated few preimage classes $F^{\boxdot}$ (with F Boolean clone), all of them are finite

Does there exist a preimage class of infinite cardinality?
Does there exist a preimage class of uncountable cardinality?

Is the lattice ${}^{\pm}\mathcal{L}_2$ of Boolean ±-preclones countable?

(compare: the lattice $\mathcal{L}_2$ of Boolean clones is countable [E.L. Post, 1921])

all what we know about Boolean ±-preclones is contained in

P. Jipsen, E. Lehtonen, and R. Pöschel, *S-preclones and the Galois connection* ${}^{S}\mathrm{Pol} - {}^{S}\mathrm{Inv}$, *Part II: Boolean ±-preclones* (in preparation, to be submitted)

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======= The classical Galois connection Pol – Inv =======

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- R. Pöschel and L.A. Kalužnin, *Funktionen- und Relationenalgebren*. Deutscher Verlag der Wissenschaften, Berlin, 1979, Birkhäuser Verlag Basel, Math. Reihe Bd. 67, 1979.

======= preclones (operads) =======

- Z. Ésik and P. Weil, *Algebraic recognizability of regular tree languages*. Theoret. Comput. Sci. 340(2), (2005), 291–321. (notion of preclone)
- E. Lehtonen, *Characterization of preclones by matrix collections*. Asian-Eur. J. Math. 3(3), (2010), 457–473.

======= Analogy to multi-sorted algebras =======

- E. Lehtonen, R. Pöschel, and T. Waldhauser, *Reflection-closed varieties of multisorted algebras and minor identities*. Algebra Universalis 79(3), (2018), Art. 70, 22 pages.

======= *S*-preclones (New) =======

- P. Jipsen, E. Lehtonen, and R. Pöschel, *S-preclones and the Galois connection* $^{S}\mathrm{Pol} - {}^{S}\mathrm{Inv}$*, Part I*, Algebra Universalis 85, 2024 (arXiv 2023: `http://arxiv.org/abs/2306.00493`).


Thank you for your ATTENTION!