Characterizing XY-homogeneity

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AAA108

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Age

In this talk we will consider only **countable relational structures**.

- The **age** of a relational structure $\mathbf{U}$ is the class (denoted by $\text{Age}(\mathbf{U})$) of all finite structures that embed into $\mathbf{U}$.

Let $\mathscr{C}$ be a class of finite structures of the same type. $\mathscr{C}$ has the:

- **Hereditary property** (HP)
  If for all $\mathbf{A} \in \mathscr{C}$, if $\mathbf{B} \hookrightarrow \mathbf{A}$, then $\mathbf{B} \in \mathscr{C}$.
- **Joint embedding property** (JEP)
  If for all $\mathbf{A}, \mathbf{B} \in \mathscr{C}$, there exists a $\mathbf{C} \in \mathscr{C}$ such that both $\mathbf{A}$ and $\mathbf{B}$ are embeddable in $\mathbf{C}$.

Characterization of ages (Fraïssé, 1953)

$\mathscr{C}$ is the age of a countable structure iff it contains, up to isomorphism, countably many structures, and it has the HP and the JEP.

$\mathbf{U}$ is called **younger** than $\mathscr{C}$ if $\text{Age}(\mathbf{U}) \subseteq \mathscr{C}$.

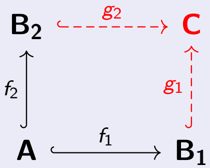
Homogeneity

Given are a relational structure **U** and an age $\mathscr{C}$.

- A **local isomorphism** of **U** is an isomorphism between finite substructures of **U**.
- **U** is **homogeneous** if every local isomorphism of **U** extends to an automorphism of **U**.

Fraïssé (1953):

$\mathscr{C}$ is the age of a countable homogeneous structure iff it has the **amalgamation property (AP)**:



$$g_1 \circ f_1 = g_2 \circ f_2$$

- Any two countable homogeneous structures of the same age are isomorphic.

Relaxing homogeneity

Cameron and Nešetřil (2002):

What happens if we relax the conditions in the definition of homogeneity?

- instead of local isomorphisms talk about
  local homomorphisms, local monomorphisms, etc.
- instead of automorphisms talk about
  endomorphisms, injective endomorphism, etc.
- XY-homogeneity
  local morphisms of type X, global morphisms of type Y

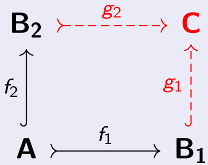
Example

MM-homogeneity: Every local monomorphism extends to a global monomorphism.

First attempt – MM-homogeneity

Cameron and Nešetřil (2002):

$\mathscr{C}$ is the age of a countable MM-homogeneous structure iff it has the **mono-amalgamation property (MAP)**:



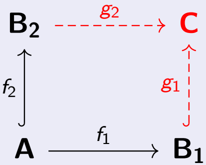
$$g_1 \circ f_1 = g_2 \circ f_2$$

- Any two countable MM-homogeneous structures **U** and **V** of the same age are **mono-equivalent**:
  every monomorphism of a finite substructure of **U** to **V** extends to a monomorphism from **U** to **V**, and vice versa.

Second attempt – HH-homogeneity

PP (2012):

$\mathscr{C}$ is the age of a countable HH-homogeneous structure iff it has the **homo-amalgamation property (HAP)**:



$$g_1 \circ f_1 = g_2 \circ f_2$$

- Any two countable homogeneous structures **U** and **V** of the same age are **homo-equivalent**:
  every homomorphism of a finite substructure of **U** to **V** extends to a homomorphism from **U** to **V**, and vice versa.

Systematization

Lockett and Truss (2014):

Consider XY-homogeneity for classes of morphisms X and Y, where $\mathrm{X} \in \{\mathrm{H}, \mathrm{M}, \mathrm{I}\}$ and $\mathrm{Y} \in \{\mathrm{H}, \mathrm{M}, \mathrm{I}, \mathrm{E}, \mathrm{B}, \mathrm{A}\}$.

Legend:

	X	Y
H	homomorphism	endomorphism
M	injective homomorphism	injective endomorphism
I	embedding	self-embedding
E	-	surjective endomorphism
B	-	bijective endomorphism
A	-	automorphism

Coleman & Co.

Coleman, Evans and Gray (2019):

$\mathscr{C}$ is the age of a countable MB-homogeneous structure iff it has both the MAP and the AMAP.

- **New**: we need two amalgamation properties!
- Building on this, Coleman (2020) proved Fraïssé-type theorems for 12 of the 18 homogeneity types:

X \Y	H	M	I	E	B	A
H	PP, ColT1	ColT1	ColT1	ColT2	ColT2	ColT2
M	?	CamNeš,ColT1	ColT1	?	ColEvGr,ColT2	ColT2
I	?	?	ColT1	?	?	Fra,ColT2

- AA handled cases IB and IM.

Goals of our study...

First goal

Find one or two main theorems which cover all fields of the table.

Second goal

Characterize the existence of structures with a set of homogeneity properties.

Third goal

Push the methods to the limit.

. . . and how to reach them

First goal

Extract the minimal set of key properties that the observed classes of morphisms have in common and that enable us to prove a Fraïssé-type theorem.

Second goal

Use topology, and, in particular, Baire category.

Third goal

Axiomatize + Avoid touching basic relations of structures in question.

Main results – non-surjective case

Let $\mathrm{Z}(\mathscr{C}) := \{f\colon \mathbf{U} \to \mathbf{V} \mid f \in \mathrm{Z}, \mathrm{Age}(\mathbf{U}), \mathrm{Age}(\mathbf{V}) \subseteq \mathscr{C}\}$.

Theorem 1

Let X and Y be $\mathscr{C}$-admissible classes.
Then $\mathscr{C}$ is the age of some countable XY-homogeneous structure if and only if there is a $\mathscr{C}$-admissible class Z such that

1. $\mathrm{X}(\mathscr{C}) \subseteq \mathrm{Z}(\mathscr{C}) \subseteq \mathrm{Y}(\mathscr{C})$, and such that
2. $\mathscr{C}$ has the ZAP.

- Moreover, any two XY-homogeneous structures **U** and **V** of the same age are **XY-equivalent**:
  every X-morphism of a finite substructure of **U** to **V** extends to a Y-morphism from **U** to **V**, and vice versa.
- Under the conditions of Theorem 1, the subset of all XY-homogeneous structures in $\mathscr{C}_{=\omega}$ with age $\mathscr{C}$ is either empty or residual.

$\mathscr{C}_{=\omega}$ is the space of all structures younger than $\mathscr{C}$ defined on ω.

Main results – surjective case

Theorem 2

Let X and Y be $\mathscr{C}$-admissible classes.
Then $\mathscr{C}$ is the age of some countable XY^{s}-homogeneous structure if and only if there is a $\mathscr{C}$-admissible class Z such that

1. $\mathrm{X}(\mathscr{C}) \subseteq \mathrm{Z}(\mathscr{C}) \subseteq \mathrm{Y}(\mathscr{C})$, and such that
2. $\mathscr{C}$ has the ZAP and the $\overline{\mathrm{Z}}\mathrm{AP}$.

- Under the conditions of Theorem 2, the subset of all XY-homogeneous structures in $\mathscr{C}_{=\omega}$ with age $\mathscr{C}$ is either empty or residual.

State of the art

X \Y	H	M	I	E	B	A
H	PP, ColT1	ColT1	ColT1	ColT2	ColT2	ColT2
M	~~?~~	CamNeš,ColT1	ColT1	~~?~~	ColEvGr,ColT2	ColT2
I	~~?~~	~~?~~	ColT1	~~?~~	~~?~~	Fra,ColT2
	Theorem 1			**Theorem 2**		

Note

Theorems 1 and 2 cover all cases from the table, but they also go beyond it.

What means admissible?

- Given a type τ of relational structures (e.g. graphs or posets...).
- $\mathscr{C}$ an age of structures of type τ.
- H designates the class of all homomorphism between structure of type τ.
- I designates the class of all embeddings between structures of type τ.
- $\mathrm{H}(\mathscr{C})$, $\mathrm{I}(\mathscr{C})$ are the restrictions of H and I to countable τ-structures younger than $\mathscr{C}$.

$\mathscr{C}$-admissible classes

A class $\mathrm{Z} \subseteq \mathrm{H}$ is called $\mathscr{C}$-admissible if

1. $\mathrm{I}(\mathscr{C}) \subseteq \mathrm{Z}$,
2. Z is closed under composition,
3. for all $h \in \mathrm{H}(\mathscr{C})$ and $\iota \in \mathrm{I}(\mathscr{C})$ if $\iota \circ h \in \mathrm{Z}$ then $h \in \mathrm{Z}$.
4. a homomorphism $h \in \mathrm{H}$ is in Z if and only if all finite restrictions of h are in Z.

Theorem 1

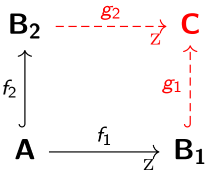
ZAP

Let $\mathscr{C}$ be an age and let $\mathrm{Z} \subseteq \mathrm{H}$. Then $\mathscr{C}$ has **the** ZAP if

for all $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathscr{C}$, $f_1 \colon \mathbf{A} \xrightarrow[\mathrm{Z}]{} \mathbf{B}_1$ and $f_2 \colon \mathbf{A} \hookrightarrow \mathbf{B}_2$,

there exist a $\mathbf{D} \in \mathscr{C}$, $g_1 \colon \mathbf{B}_1 \hookrightarrow \mathbf{D}$ and $g_2 \colon \mathbf{B}_2 \xrightarrow[\mathrm{Z}]{} \mathbf{D}$

such that $g_1 \circ f_1 = g_2 \circ f_2$:



Theorem 1

The class $\overline{\mathrm{Z}}$

Let $\mathbf{A}$ and $\mathbf{B}$ be structures of the same type.

- A **multimorphism** $\bar{f}\colon \mathbf{A} \multimap \mathbf{B}$ is a quadruple $(\mathbf{A}, f, \mathbf{S}, \mathbf{B})$, where
  - $\mathbf{S} \leq \mathbf{B}$, and
  - $f\colon \mathbf{S} \twoheadrightarrow \mathbf{A}$.

$$\mathbf{A} \xrightarrow{\bar{f}}\!\!\circ\; \mathbf{B} \quad \text{corresponds to} \quad \mathbf{A} \xleftarrow{f}\!\!\!\!\leftarrow \mathbf{S} \xhookrightarrow{=} \mathbf{B}.$$

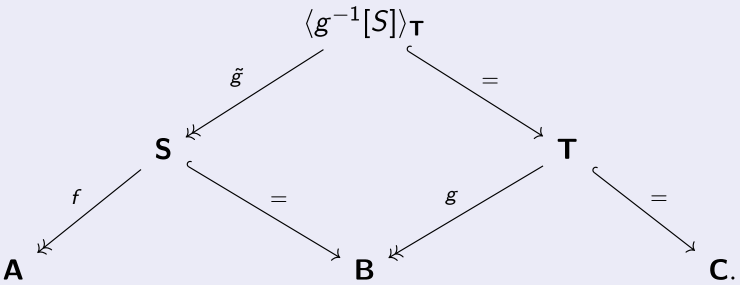
- The structure $\mathbf{S}$ is called the **image** of $\bar{f}$ and is denoted by $\operatorname{im}(\bar{f})$.
- If Z is a $\mathscr{C}$-admissible class, then we say that $\bar{f}$ is of type Z if $f \in \mathrm{Z}$.
- The class of all multimorphisms of type Z is denoted by $\overline{\mathrm{Z}}$.

$\bar{f}$ is essentially just the inverse of the surjective partial homomorphism $f\colon \mathbf{B} \multimap\!\!\rightarrow \mathbf{A}$.

Multimorphisms may be composed

Let

- $\bar{f} : \mathbf{A} \multimap \mathbf{B}$, where $\bar{f} = (\mathbf{A}, f, \mathbf{S}, \mathbf{B})$,
- $\bar{g} : \mathbf{B} \multimap \mathbf{C}$, where $\bar{g} = (\mathbf{B}, g, \mathbf{T}, \mathbf{C})$, and
- $\tilde{g} : \langle g^{-1}[S] \rangle_{\mathbf{T}} \twoheadrightarrow \mathbf{S}$ the restriction of g to $g^{-1}[S]$.



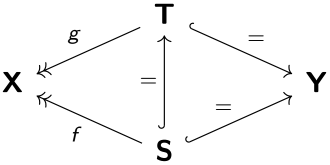
Then $\bar{g} \circ \bar{f} := (\mathbf{A}, f \circ \tilde{g}, \langle g^{-1}[S] \rangle_{\mathbf{T}}, \mathbf{C})$.

Multimorphisms are partially ordered

Let

- $\bar{f}: \mathbf{X} \multimap \mathbf{Y}$, where $\bar{f} = (\mathbf{X}, f, \mathbf{S}, \mathbf{Y})$, and
- $\bar{g}: \mathbf{X} \multimap \mathbf{Y}$, where $\bar{g} = (\mathbf{X}, g, \mathbf{T}, \mathbf{Y})$

Then $\bar{f} \leq \bar{g}$ if $\mathbf{S} \leq \mathbf{T}$, and if the following diagram commutes:



The composition operation is monotonous in both arguments, i.e.,

$$\bar{f}_1 \leq \bar{f}_2 \wedge \bar{g}_1 \leq \bar{g}_2 \implies \bar{g}_1 \circ \bar{f}_1 \leq \bar{g}_2 \circ \bar{f}_2.$$

The category of structures with multimorphisms forms a so-called locally partially ordered 2-category.

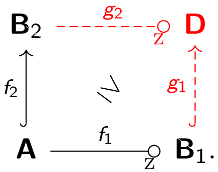
$\overline{\mathrm{Z}}\mathrm{AP}$

Let $\mathscr{C}$ be an age and let Z be a $\mathscr{C}$-admissible class. Then $\mathscr{C}$ has **the $\overline{\mathrm{Z}}\mathrm{AP}$** if

for all $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathscr{C}$, $f_1 \colon \mathbf{A} \xrightarrow[\mathrm{Z}]{} \mathbf{B}_1$ and $f_2 \colon \mathbf{A} \hookrightarrow \mathbf{B}_2$,

there exist a $\mathbf{D} \in \mathscr{C}$, $g_1 \colon \mathbf{B}_1 \hookrightarrow \mathbf{D}$ and $g_2 \colon \mathbf{B}_2 \xrightarrow[\mathrm{Z}]{} \mathbf{D}$

such that $g_1 \circ f_1 \leq g_2 \circ f_2$:



Theorem 2