The semigroup topologies of the symmetric inverse monoid

J. D. Mitchell

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Algebra vs topology

There are many algebraic objects that are naturally endowed with a topological structure:

- ★ normed vector spaces; groups: the additive groups $\mathbb{R}^n$, $n \in \mathbb{N}$
- ★ the automorphism groups $\mathrm{Aut}(X)$ of relational structures X: $(\mathbb{N}, =)$, the random graph R, $\mathbb{Q}$, ...
- ★ the endomorphism monoids $\mathrm{End}(X)$ of relational structures X: the random graph, $\mathbb{Q}$, ...
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In these examples, the algebraic operations are compatible with the topological structure, i.e. they are ***continuous***.

Question

Given an algebraic object what are the compatible topologies?

Group topology

A topology on a group G is called a ***group topology*** if $G \times G \to G$ defined by

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and $G \to G$ defined by

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Examples:

★ The discrete or trivial topology on any group.

★ The standard topology on the additive group of real numbers $\mathbb{R}$.

★ The symmetric group $\mathrm{Sym}(\mathbb{N})$ with the subspace topology inherited from $\mathbb{N}^{\mathbb{N}}$.

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Non-topologizable groups

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Does there exist an infinite group whose only group topologies are the trivial and discrete topologies?

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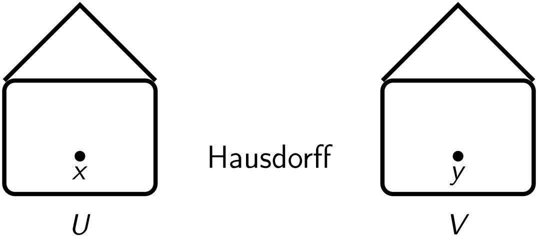
Yes: in ZFC (Ol'shanskij, '80)

So, not every group can have a meaningful topology.

1 group topology

Unique Hausdorff topologies:



- ★ Finite rings, fields, or groups (discrete topology)
- ★ Every finite dimensional real or complex vector space
- ★ The general linear groups $\mathrm{GL}(n, \mathbb{R})$
- ★ $\cdots$

∞ many group topologies

Example

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- ★ The standard topology on $(\mathbb{R}^n, +)$ is a group topology.
- ★ $(\mathbb{R}, +)$ is isomorphic as a group to $(\mathbb{R}^n, +)$ for all $n \in \mathbb{N}$.

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- ★ The standard topology on $(\mathbb{R}^n, +)$ is a group topology.
- ★ $(\mathbb{R}, +)$ is isomorphic as a group to $(\mathbb{R}^n, +)$ for all $n \in \mathbb{N}$.
- ★ $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ if and only if $m = n$. $\square$

Polish groups

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- ★ the additive group $\mathbb{R}$
- ★ the symmetric group $\mathrm{Sym}(\mathbb{N})$
- ★ the automorphism group of any countable relational structure, $(\mathbb{Q}, <)$, the random graph, and so on
- ★ . . .

0 or 1 Polish group topologies

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Theorem (Dudley '61)

If G is a completely metrizable group and F is a free group with the discrete topology, then any homomorphism from G to F is continuous.

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It is consistent with ZF without C that every Polish group has a unique Polish group topology.

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Remember the example of the additive group of reals $\mathbb{R}$!

The Baire space

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The Baire space is Polish: the metric

$$d(f, g) = \begin{cases} \frac{1}{n} & \text{if } f \neq g \text{ and } n = \min\{m \in \mathbb{N} : (m)f \neq (m)g\} \\ 0 & \text{if } f = g \end{cases}$$

is complete, and the eventually constant functions are countable and dense.

Symmetric group

Theorem (Folklore)

If G is a Polish group with respect to topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ and $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then $\mathcal{T}_1 = \mathcal{T}_2$.

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Corollary

The pointwise topology is the unique Polish group topology on $\operatorname{Sym}(\mathbb{N})$.

Semigroup topologies

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Examples of topological semigroups:

★ topological groups

★ the lower limit topology on $(\mathbb{R}, +)$ (not a group topology, not Polish)

★ the endomorphism monoid $\text{End}(X)$ where X is any relational structure over a countable set (such as a graph, for example)

★ the monoid $C(X)$ of continuous functions from a compact metrizable space X to itself.

The full transformation monoid

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Every semigroup embeds into X^X for some set X.

$\mathbb{N}^{\mathbb{N}}$ is a Polish semigroup with the topology of the Baire space (the pointwise topology).

0 or 1 Polish semigroup topologies

Elliott, Jonušas, Mesyan, JDM, Morayne, Péresse '23

The following monoids have no Polish semigroup topology:

- ★ the monoid of binary relations on $\mathbb{N}$;
- ★ diagram monoids (the partition monoid, dual symmetric inverse monoid);

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The following monoids have a unique Polish semigroup topology:

★ the full transformation monoid $\mathbb{N}^{\mathbb{N}}$ (the pointwise topology)

★ continuous functions on the Hilbert cube $[0,1]^{\mathbb{N}}$, and the Cantor set $2^{\mathbb{N}}$ (the compact-open topology)

★ (Elliott, Jonušas, JDM, Péresse, Pinsker '23) the endomorphism monoids of the random graph R; the rationals $(\mathbb{Q}, \leq)$, ... (the pointwise topology)

★ (Pinsker, Schindler '23) the increasing functions on $\mathbb{Q}$ (the pointwise topology)

★ ...

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for all $a \in \mathbb{N}$ and $n \in \omega + 1$.

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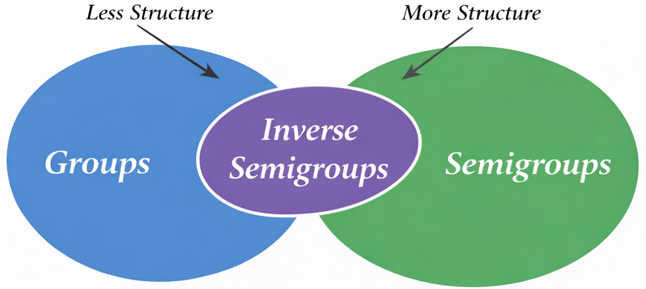
There are infinitely many more Polish semigroup topologies between these two.

Inverse semigroups

If S is a semigroup and $x \in S$, then $y \in S$ is an ***inverse*** of x if $xyx = x$ and $yxy = y$.

A semigroup S where every $x \in S$ has a unique inverse $x^{-1} \in S$, is called an ***inverse semigroup***.



Examples:

★ semilattices

★ the bicyclic monoid 🚲 $= \langle b, c \mid bc = 1 \rangle$

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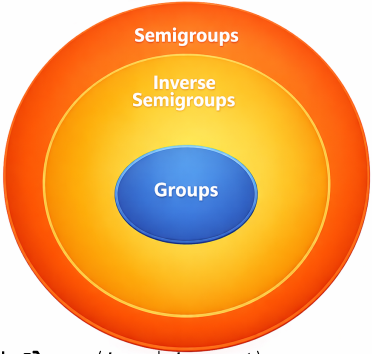

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Symmetric inverse monoid

If X is any set, then the set I_X of bijections between subsets of X is an inverse semigroup under the usual composition of binary relations:

$$f \circ g = \{(x, z) \in X \times X : (x, y) \in f, (y, z) \in g, y \in X\}$$

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Example:

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ - & 2 & 4 & - \end{pmatrix}, \qquad f^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ - & 2 & - & - \end{pmatrix}$$

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Try extending the pointwise topology from $\mathrm{Sym}(\mathbb{N})$ to $I_{\mathbb{N}}$?

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The bad: I_0 is not T_1.

The least T_1 shift-continuous topology

If $\lambda_s, \rho_s : S \to S$ defined by

$$(x)\lambda_s = sx \quad \text{and} \quad (x)\rho_s = xs$$

are continuous for all $s \in S$, then S is ***shift-continuous***.

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Can we find the least T_1 shift-continuous topology for $I_{\mathbb{N}}$?

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- ★ So $\{f \in I_{\mathbb{N}} : (m, n) \in f\}$ and $\{f \in I_{\mathbb{N}} : (m, n) \notin f\}$ are open.

Properties of I_1

Topology I_1

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Theorem (Elliott, Jonušas, JDM, Mesyan, Morayne, Péresse '23)

The topology I_1 on $I_{\mathbb{N}}$ is

- ★ *Polish and compact(!?);*
- ★ *the least T_1 shift continuous topology;*
- ★ *not a semigroup topology but inversion is continuous.*

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Can we find a T_1 (or higher) semigroup topology for $I_{\mathbb{N}}$?

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★ For $f \in I_{\mathbb{N}}$ define $f' \in \mathbb{N}^{\mathbb{N}}$ by

$$(x+1)f' = \begin{cases} (x)f + 1 & \text{if } x \in \operatorname{dom}(f) \\ 0 & \text{otherwise} \end{cases}$$

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★ The pointwise topology on $\mathbb{N}^{\mathbb{N}}$ induces a semigroup topology I_2 on $I_{\mathbb{N}}$ via this embedding.

A semigroup topology

The topology $\mathcal{I}_2$

The topology with sub-basic sets

$$\{f \in I_{\mathbb{N}} : (m, n) \in f\} \quad \& \quad \{f \in I_{\mathbb{N}} : m \notin \text{dom}(f)\}.$$

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We get immediately from the definition:

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★ inversion is not continuous!

Properties of I_2 and I_3

I_2 has a dual $I_3 = I_2^{-1} = \{U^{-1} : U \in I_2\}$ where $U^{-1} = \{f^{-1} : f \in U\}$.

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Theorem (Elliott, Jonušas, JDM, Mesyan, Morayne, Péresse '23)

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★ $I_1 \subsetneq I_2 \cap I_3$

The Polish inverse semigroup topology for $I_{\mathbb{N}}$

The topology I_4

I_4 is generated by $I_2 \cup I_3$ and has sub-basic sets

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Theorem (Elliott, Jonušas, JDM, Mesyan, Morayne, Péresse '23)

The topology I_4 on $I_{\mathbb{N}}$ is:

- ★ *the unique T_1 and second-countable inverse semigroup topology*
- ★ *the maximal second-countable semigroup topology*

A topology for every function

Suppose that $f : \omega + 1 \to \omega + 1$ is any function. We define $\mathcal{T}_f$ to be the least topology on $I_{\mathbb{N}}$ containing I_2 and the sets

$$U_{f,n,X} = \{g \in I_{\mathbb{N}} : |\operatorname{im}(g) \setminus X| \geq n \text{ and } |X \cap \operatorname{im}(g)| \leq (n)f\}$$

for all $n \in \mathbb{N}$ and $X \subseteq \mathbb{N}$ is finite.

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An element $g \in U_{f,n,X}$ must map at least n points outside of X and is allowed at most $(n)f$ mistakes, i.e. points in the image of g in X.

Waning

Waning

Waning function

We say that a non-increasing function $f : \omega + 1 \to \omega + 1$ is ***waning*** if:

★ f is constant with value ω; or

★ $(j+1)f < (j)f$ for all $j \in \omega$ such that $0 \neq (j)f \in \omega$.

If f is a waning function, continuing to speak roughly, partial functions in $U_{f,n,X}$ defined on more points are allowed fewer mistakes.

Classification of semigroup topologies

Theorem (Bardyla, Elliott, JDM, Péresse '24)

If $\mathcal{T}$ is a T_1 second countable semigroup topology on $I_{\mathbb{N}}$, then there exists a waning function f such that either $\mathcal{T} = \mathcal{T}_f$, or $\mathcal{T}^{-1} = \mathcal{T}_f$.
Conversely, if f and g are distinct waning functions, then $\mathcal{T}_f$ and $\mathcal{T}_g$ are distinct Polish semigroup topologies for $I_{\mathbb{N}}$.

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Lemma (Bardyla, Elliott, JDM, Péresse '24)

If f and g are waning functions, then $\mathcal{T}_f \subseteq \mathcal{T}_g$ if and only if $(n)g \leq (n)f$ for all $n \in \omega$.

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Corollary (Bardyla, Elliott, JDM, Péresse '24)

The partial order of Polish semigroup topologies on $I_{\mathbb{N}}$ contains:

(a) *infinite descending chains;*

(b) *finite but not infinite ascending chains; and*

(c) *every finite partial order.*

Thanks!

Topology is relevant