Lagrange-like interpolation in unitary rings, Boolean algebras and Boolean posets

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Support of the research of the first author by the Czech Science Foundation (GAČR), project 25-20013L, and by IGA, project PřF 2025 008, and support of the research of the second author by the Austrian Science Fund (FWF), project 10.55776/PIN5424624, is gratefully acknowledged.

108th Workshop on General Algebra, TU Wien, Austria, February 5, 2026

Outline:

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- Interpolation in fields

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Interpolation in fields

Classical Lagrange interpolation

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Theorem 1

Let $\mathbf{K} = (K, +, \cdot)$ *be a field,* $n > 1$*,* $a_1, \ldots, a_n$ *different elements of* K *and* $f\colon K \to K$ *and define* $p_1, \ldots, p_n, p\colon K \to K$ *by*

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$$p_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - a_j}{a_i - a_j} \text{ for } i = 1, \ldots, n,$$

$$p(x) := \sum_{i=1}^{n} f(a_i) p_i(x)$$

for all $x \in K$.

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Here and in the following $p_i(a_k) = \delta_{ik}$ for $i, k = 1, \ldots, n$.

Interpolation in unitary rings

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Theorem 2

Let $(R, +, \cdot, 0, 1)$ *be a unitary ring,* $n > 1$*,* $a_1, \ldots, a_n$ *different elements of* R *and* $f\colon R \to R$ *and define* $\Delta, p_1, \ldots, p_n, p\colon R \to R$ *by*

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Let $(B, \vee, \wedge, ', 0, 1)$ *be a Boolean algebra,* $n > 1$*,* $a_1, \ldots, a_n$ *different elements of* B *and* $f\colon B \to B$ *and define* $\Delta, p_1, \ldots, p_n, p\colon B \to B$ *by*

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For $k, j = 1, \ldots, n$ we have $(a_k' \wedge a_j) \vee (a_k \wedge a_j') = 0$ if and only if $k = j$.

Interpolation in Boolean posets

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A distributive poset is a poset $(P, \leq)$ satisfying the LU-identity

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A lattice is distributive if and only if it is a distributive poset.

Boolean posets

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Definition 5

A *poset with complementation* *is a bounded poset* $(P, \leq, ', 0, 1)$ *with a unary operation* $'$ *satisfying the LU-identities*

$$U(x, x') \approx 1 \text{ and } L(x, x') \approx 0.$$

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A *Boolean poset* *is a distributive poset with complementation.*

In the following we identify singletons with their unique element. For a subset A of a poset we denote by Min A the set of its minimal elements.

Example of a Boolean poset

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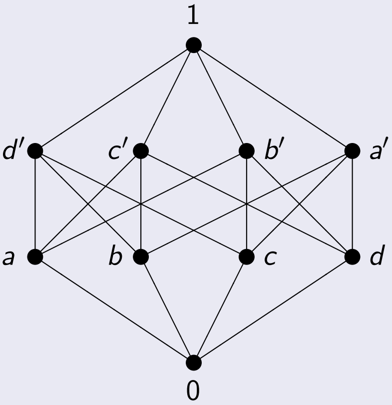
Example 6



(with $0' = 1$, $1' = 0$ *and involution* $'$*) is a Boolean poset.*

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Theorem 7

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for all $A\subseteq B$ *and all* $x\in B$*.*

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Final remark

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For $k, j = 1, \ldots, n$ we have $\operatorname{Min} U\big(L(a'_k, a_j), L(a_k, a'_j)\big) = 0$ if and only if $k = j$.

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Thank you for your attention!