An axiomatization of double Boolean algebras

Leonard Kwuida

Bern University of Applied Sciences

joint work with:
Prosenjit Howlader: Institute of Information Science, Academia Sinica
Churn-Jung Liau: Institute of Information Science, Academia Sinica
Mike Behrisch: Technische Universität Wien, Austria.

Funded by SNF Grant IZSEZO_233403/1: Axiomatizations of Double Boolean Algebras

1. Preliminaries
2. Example of a dBa
3. Minimal set of Axioms

Double Boolean algebra [2]

A *double Boolean algebra* (dBa) is an algebra $\mathbf{D} := (D; \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$ of type (2,2,1,1,0,0) which satisfies the equations (1a.) – (11a.), (1b.) – (11b.) and (12.).

Idempotence:

1a. $(x \sqcap x) \sqcap y = x \sqcap y$ 1b. $(x \sqcup x) \sqcup y = x \sqcup y$

8a. $\neg(x \sqcap x) = \neg x$ 8b. $\lrcorner(x \sqcup x) = \lrcorner x$

Commutative and associative:

2a. $x \sqcap y = y \sqcap x$ 2b. $x \sqcup y = y \sqcup x$

3a. $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$ 3b. $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$

Absorption:

4a. $x \sqcap (x \sqcup y) = x \sqcap x$ 4b. $x \sqcup (x \sqcap y) = x \sqcup x$

5a. $x \sqcap (x \vee y) = x \sqcap x$ 5b. $x \sqcup (x \wedge y) = x \sqcup x$

Distributive:

6a. $x \sqcap (y \vee z) = (x \sqcap y) \vee (x \sqcap z)$ 6b. $x \sqcup (y \wedge z) = (x \sqcup y) \wedge (x \sqcup z)$

Axioms for negations:

7a. $\neg\neg(x \sqcap y) = x \sqcap y$ 7b. $\lrcorner\lrcorner(x \sqcup y) = x \sqcup y$

9a. $x \sqcap \neg x = \bot$ 9b. $x \sqcup \lrcorner x = \top$

where $x \vee y := \neg(\neg x \sqcap \neg y)$ and $x \wedge y := \lrcorner(\lrcorner x \sqcup \lrcorner y)$.

Axioms for constants:

10a. $\neg\bot = \top \sqcap \top$ 10b. $\lrcorner\top = \bot \sqcup \bot$

11a. $\lrcorner\bot = \top$ 11b. $\neg\top = \bot$

Proto's axiom:

12. $(x \sqcap x) \sqcup (x \sqcap x) = (x \sqcup x) \sqcap (x \sqcup x)$.

- A weak dBa satisfies equations (1a.) – (11a.) and duals (1b.) – (11b.)

- The relation $\sqsubseteq$ on **D**, defined below, is a quasi-order.

$$x \sqsubseteq y :\iff \quad x \sqcap y = x \sqcap x \text{ and } x \sqcup y = y \sqcup y.$$

- $D_\sqcap := \{x \in D \mid x \sqcap x = x\}$ and $D_\sqcup := \{x \in D \mid x \sqcup x = x\}$
- $\mathbf{D}_\sqcap := (D_\sqcap; \sqcap, \vee, \neg, \bot, \top \sqcap \top)$ and $\mathbf{D}_\sqcup := (D_\sqcup; \wedge, \sqcup, \lrcorner, \bot \sqcup \bot, \top)$ are Boolean algebras.

Contexts, Concepts and Protoconcepts [2]

- $\mathbb{K} := (G, M, R)$ with $R \subseteq G \times M$ is called a formal context (or polarity).
- Concept forming operator. For $A \subseteq G, B \subseteq M$, define

  $$A' := \{m \in M \mid gRm\ \forall g \in A\} \quad \text{and} \quad B' := \{g \in G \mid gRm\ \forall m \in B\}$$

- Concepts and friends: A pair (A, B) is a
  - concept if $A' = B$ and $B' = A$
  - semiconcept of if $A' = B$ or $B' = A$.
  - protoconcept if $A'' = B'$.
  - preconcept if $A \subseteq B'$
- $\mathfrak{V}(\mathbb{K})$, $\mathfrak{P}(\mathbb{K})$, $\mathfrak{H}(\mathbb{K})$, $\mathfrak{B}(\mathbb{K})$ denotes the set of pre-, proto-, semi- and concepts of $\mathbb{K}$, respectively.
- $\mathfrak{B}(\mathbb{K}) \subseteq \mathfrak{H}(\mathbb{K}) \subseteq \mathfrak{P}(\mathbb{K}) \subseteq \mathfrak{V}(\mathbb{K})$.

Algebra of Protoconcepts [2]

- Logical operator on concepts: (meet and join)

$$(A,B)\wedge(C,D) = (A\cap C,(B\cup D)'') = (A\cap C,(A\cap C)')$$
$$(A,B)\vee(C,D) = ((A\cup C)'',B\cap D) = ((B\cap D)',B\cap D)$$

- Logical operators on preconcepts: For (A,B) and (C,D) in $\mathfrak{V}(\mathbb{K})$:

$$(A,B)\sqcap(C,D) := (A\cap C,(A\cap C)')$$
$$(A,B)\sqcup(C,D) := ((B\cap D)',B\cap D)$$
$$\neg(A,B) := (G\setminus A,(G\setminus A)')$$
$$\lrcorner(A,B) := ((M\setminus B)',M\setminus B)$$
$$\top := (G,G') \quad \text{and} \quad \bot := (M',M).$$

- $\underline{\mathfrak{P}}(\mathbb{K}) := (\mathfrak{P}(\mathbb{K}),\sqcup,\sqcap,\neg,\lrcorner,\top,\bot)$ is a double Boolean algebra.
- DBas generate the equational theory of protoconcept algebras [2].

Motivation for the present work

- The definition of dBas given by Rudolf Wille contains a large number of axioms $(2 \times 11 + 1)$.
- Immediate Goal: Find a non redundant and complete subset of Wille's set of dBas axioms.
- Further Goal: Find alternative sets of axioms for dBas.

A small subset of Wille's dBas axioms

- $\mathbf{D} := (D; \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$ is called a D-core algebra if it satisfies

Commutative:

2a. $x \sqcap y = y \sqcap x$ 2b. $x \sqcup y = y \sqcup x$

Idempotence:

4a. $\neg(x \sqcap x) = \neg x$ 4b. $\lrcorner(x \sqcup x) = \lrcorner x$

Axioms for negations:

7a. $\neg\neg(x \sqcap y) = x \sqcap y$ 7b. $\lrcorner\lrcorner(x \sqcup y) = x \sqcup y$

9a. $x \sqcap \neg x = \bot$ 9b. $x \sqcup \lrcorner x = \top$

Absorption:

4a. $x \sqcap (x \sqcup y) = x \sqcap x$ 4b. $x \sqcup (x \sqcap y) = x \sqcup x$

Distributive:

6a. $x \sqcap (y \vee z) = (x \sqcap y) \vee (x \sqcap z)$ 6b. $x \sqcup (y \wedge z) = (x \sqcup y) \wedge (x \sqcup z)$

12. $(x \sqcap x) \sqcup (x \sqcap x) = (x \sqcup x) \sqcap (x \sqcup x)$.

Main Result

An algebra $(D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$ is a dBa if and only if it is a **D-core** algebra.

D-core [1]

Definition 1

An algebra $\mathbf{D} := (D; \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$ satisfying the following properties is called a *D-core algebra*. For any $x, y, z \in D$,

$(1a)$ $x \sqcap y = y \sqcap x$	$(1b)$ $x \sqcup y = y \sqcup x$
$(2a)$ $\neg(x \sqcap x) = \neg x$	$(2b)$ $\lrcorner(x \sqcup x) = \lrcorner x$
$(3a)$ $x \sqcap (x \sqcup y) = x \sqcap x$	$(3b)$ $x \sqcup (x \sqcap y) = x \sqcup x$
$(4a)$ $x \sqcap (y \vee z) = (x \sqcap y) \vee (x \sqcap z)$	$(4b)$ $x \sqcup (y \wedge z) = (x \sqcup y) \wedge (x \sqcup z)$
$(5a)$ $\neg\neg(x \sqcap y) = x \sqcap y$	$(5b)$ $\lrcorner\lrcorner(x \sqcup y) = x \sqcup y$
$(6a)$ $x \sqcap \neg x = \bot$	$(6b)$ $x \sqcup \lrcorner x = \top$

(7) $(x \sqcap x) \sqcup (x \sqcap x) = (x \sqcup x) \sqcap (x \sqcup x)$

In the rest of this presentation, $\mathbf{D}$:=$(D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$ is a D-core algebra, and $x, y, z, \cdots \in D$.

Elimination of Wille's 1a., 1b.

Proposition 2

$(1a)\ \neg\neg x = x \sqcap x$

$(2a)\ (x \sqcap y) \sqcap (x \sqcap y) = x \sqcap y$

$(3a)\ \neg(x \sqcap (y \vee z)) = \neg(x \sqcap y) \sqcap \neg(x \sqcap z)$

$(4a)\ x \vee x = x \sqcap x$

$(5a)\ \neg\neg\neg x = \neg x$

$(6a)\ \neg x \sqcap \neg x = \neg x$

$(1b)\ \lrcorner\lrcorner x = x \sqcup x$

$(2b)\ (x \sqcup y) \sqcup (x \sqcup y) = x \sqcup y$

$(3b)\ \lrcorner(x \sqcup (y \wedge z)) = \lrcorner(x \sqcup y) \sqcup \lrcorner(x$

$(4b)\ x \wedge x = x \sqcup x$

$(5b)\ \lrcorner\lrcorner\lrcorner x = \lrcorner x$

$(6b)\ \lrcorner x \sqcup \lrcorner x = \lrcorner x$

$$(x \sqcap x) \sqcap y \overset{\text{D.1 }(1a)}{=} y \sqcap (x \sqcap x) \overset{\text{P.2 }(4a)}{=} y \sqcap (x \vee x) \overset{\text{D.1 }(4a)}{=} (y \sqcap x) \vee (y \sqcap x)$$

$$\overset{\text{P.2 }(4a)}{=} (y \sqcap x) \sqcap (y \sqcap x) \overset{\text{P.2 }(2a)}{=} y \sqcap x \overset{\text{D.1 }(1a)}{=} x \sqcap y.$$

Elimination of Wille's 1a., 1b.

Proposition 2

$(1a)\ \neg\neg x = x \sqcap x$

$(2a)\ (x \sqcap y) \sqcap (x \sqcap y) = x \sqcap y$

$(3a)\ \neg(x \sqcap (y \vee z)) = \neg(x \sqcap y) \sqcap \neg(x \sqcap z)$

$(4a)\ x \vee x = x \sqcap x$

$(5a)\ \neg\neg\neg x = \neg x$

$(6a)\ \neg x \sqcap \neg x = \neg x$

$(1b)\ \lrcorner\lrcorner x = x \sqcup x$

$(2b)\ (x \sqcup y) \sqcup (x \sqcup y) = x \sqcup y$

$(3b)\ \lrcorner(x \sqcup (y \wedge z)) = \lrcorner(x \sqcup y) \sqcup \lrcorner(x$

$(4b)\ x \wedge x = x \sqcup x$

$(5b)\ \lrcorner\lrcorner\lrcorner x = \lrcorner x$

$(6b)\ \lrcorner x \sqcup \lrcorner x = \lrcorner x$

$$(x \sqcap x) \sqcap y \overset{\text{D.1 }(1a)}{=} y \sqcap (x \sqcap x) \overset{\text{P.2 }(4a)}{=} y \sqcap (x \vee x) \overset{\text{D.1 }(4a)}{=} (y \sqcap x) \vee (y \sqcap x)$$

$$\overset{\text{P.2 }(4a)}{=} (y \sqcap x) \sqcap (y \sqcap x) \overset{\text{P.2 }(2a)}{=} y \sqcap x \overset{\text{D.1 }(1a)}{=} x \sqcap y.$$

Elimination of Wille's 10a., 11a, and 10b., 11b.

Theorem 3

$(1a)\ (x \sqcap x) \sqcap y = x \sqcap y$	$(1b)\ (x \sqcup x) \sqcup y = x \sqcup y$
$(2a)\ \neg\top = \bot$	$(2b)\ \lrcorner\bot = \top$
$(3a)\ \neg\bot = \top \sqcap \top$	$(3b)\ \lrcorner\top = \bot \sqcup \bot$

$$\text{(2a)}\quad \neg\top \overset{\text{P2 }(5a)}{=} \neg\neg\neg\top \overset{\text{P2 }(1a)}{=} \neg\top \sqcap \neg\top \overset{\text{D1 }(3a)}{=} \neg\top \sqcap (\neg\top \sqcup \lrcorner\neg\top)$$

$$\overset{\text{D1 }(6b)}{=} \neg\top \sqcap \top \overset{\text{D1 }(6a)}{=} \bot.$$

$$\text{(3a)}\quad \top \sqcap \top \overset{\text{P2 }(1a)}{=} \neg\neg\top \overset{\text{T3 }(2a)}{=} \neg\bot.$$

De Morgan laws

Proposition 4

$$(1a)\ \neg(x \sqcap y) = \neg x \vee \neg y \quad (1b)\ \lrcorner(x \sqcup y) = \lrcorner x \wedge \lrcorner y$$
$$(2a)\ \neg(x \vee y) = \neg x \sqcap \neg y \quad (2b)\ \lrcorner(x \wedge y) = \lrcorner x \sqcup \lrcorner y$$

Join and meet with top/bottom

Proposition 5

$(1a)\ x \sqcap \top = x \sqcap x$	$(1b)\ x \sqcup \bot = x \sqcup x$
$(2a)\ \neg x \sqcap \top = \neg x$	$(2b)\ \lrcorner x \sqcup \bot = \lrcorner x.$
$(3a)\ (x \sqcap y) \sqcap \top = x \sqcap y$	$(3b)\ (x \sqcup y) \sqcap \top = x \sqcup y$
$(4a)\ \neg x \sqcap \neg \bot = \neg x$	$(4b)\ \lrcorner x \sqcup \lrcorner \top = \lrcorner x.$
$(5a)\ x \vee \bot = x \sqcap x$	$(5b)\ x \wedge \top = x \sqcup x$
$(6a)\ \neg x \vee \bot = \neg x$	$(6b)\ \lrcorner x \wedge \top = \lrcorner x$
$(7a)\ (x \sqcap y) \vee \bot = x \sqcap y$	$(7b)\ (x \sqcup y) \wedge \top = x \sqcup y$
$(8a)\ x \sqcap (y \vee \top) = x \sqcap (x \vee y)$	$(8b)\ x \sqcup (y \wedge \bot) = x \sqcup (x \wedge y)$
$(9a)\ \bot \sqcap \bot = \bot$	$(9b)\ \top \sqcup \top = \top$
$(10a)\ (\bot \sqcap \neg x) \sqcap x = \bot$	$(10b)\ (\top \sqcup \lrcorner x) \sqcup x = \top$
$(11a)\ (\bot \sqcap x) \sqcap \neg x = \bot$	$(11b)\ (\top \sqcup x) \sqcup \lrcorner x = \top$
$(12a)\ \bot \sqcap \lrcorner x = \bot$	$(12b)\ \top \sqcup \neg x = \top$
$(13a)\ \bot \sqcap \neg \lrcorner x = \bot$	$(13b)\ \top \sqcup \lrcorner \neg x = \top.$

Bottom

$\bot \sqcap x = \bot$ for $x \in \{\bot, \lrcorner x, \neg \lrcorner x\}$. What about $\bot \sqcap x$ in general?

Proposition 6

$(1a)$ $x \sqcup (\bot \sqcup y) = x \sqcup y$

$(2a)$ $(\bot \sqcap x) \sqcup y = \bot \sqcup y$

$(3a)$ $(\bot \sqcap x) \sqcap (\bot \sqcup y) = \bot \sqcap x$

$(4a)$ $(\bot \sqcap x) \sqcap \neg(x \sqcap \neg y) = (\bot \sqcap x) \sqcap y$

$(1b)$ $x \sqcap (\top \sqcap y) = x \sqcap y$

$(2b)$ $(\top \sqcup x) \sqcap y = \top \sqcap y$

$(3b)$ $(\top \sqcup x) \sqcup (\top \sqcap y) = \top \sqcup x$

$(4b)$ $(\top \sqcup x) \sqcup \lrcorner(x \sqcup \lrcorner y) = (\top \sqcup x) \sqcup$

Proposition 7

$(1a)$ $\bot \sqcap \neg(\neg x \sqcap \lrcorner y) = \bot \sqcap x$

$(2a)$ $x \sqcap \neg(\neg x \sqcap \neg y) = x \sqcap \neg(\neg y \sqcap \neg \top)$

$(3a)$ $x \sqcap \neg(\neg y \sqcap \neg(x \sqcup z)) = x \sqcap \neg(\bot \sqcap \neg y)$

$(4a)$ $\bot \sqcap x = \bot$

$(1b)$ $\top \sqcup \lrcorner(\lrcorner x \sqcup \neg y) = \top \sqcup x$

$(2b)$ $x \sqcup \lrcorner(\lrcorner x \sqcup \lrcorner y) = x \sqcup \lrcorner(\lrcorner y \sqcup \lrcorner \bot$

$(3b)$ $x \sqcup \lrcorner(\lrcorner y \sqcup \lrcorner(x \sqcap z)) = x \sqcup \lrcorner($

$(4b)$ $\top \sqcup x = \top.$

Elimination of Wille's 6a. and 6b.

Theorem 8

$(1a)\ x \sqcap (x \vee y) = x \sqcap x \quad (1b)\ x \sqcup (x \wedge y) = x \sqcup x.$

$$
\begin{aligned}
x \sqcap \neg(\neg x \sqcap \neg y) &\overset{\text{P. 7(2a)}}{=} x \sqcap \neg(\neg\top \sqcap \neg y) \overset{\text{T. 3(2a)}}{=} x \sqcap \neg(\bot \sqcap \neg y) \\
&\overset{\text{P. 7(3a)}}{=} x \sqcap \neg(\neg y \sqcap \neg(x \sqcup \top)) \overset{\text{P. 7(4b)}}{=} x \sqcap \neg(\neg y \sqcap \neg\top) \\
&\overset{\text{T. 3(2a)}}{=} x \sqcap \neg(\neg y \sqcap \bot) \overset{\text{P. 7(4a)}}{=} x \sqcap \neg\bot \\
&\overset{\text{T. 3(3a)}}{=} x \sqcap (\top \sqcap \top) \overset{\text{T. 3(1a)}}{=} x \sqcap \top \overset{\text{P.5 } (1a)}{=} x \sqcap x
\end{aligned}
$$

Elimination of Wille's 6a. and 6b.

Theorem 8

$(1a)\ x \sqcap (x \vee y) = x \sqcap x \quad (1b)\ x \sqcup (x \wedge y) = x \sqcup x.$

$$
\begin{aligned}
x \sqcap \neg(\neg x \sqcap \neg y) &\overset{\text{P. 7(2a)}}{=} x \sqcap \neg(\neg\top \sqcap \neg y) \overset{\text{T. 3(2a)}}{=} x \sqcap \neg(\bot \sqcap \neg y) \\
&\overset{\text{P. 7(3a)}}{=} x \sqcap \neg(\neg y \sqcap \neg(x \sqcup \top)) \overset{\text{P. 7(4b)}}{=} x \sqcap \neg(\neg y \sqcap \neg\top) \\
&\overset{\text{T. 3(2a)}}{=} x \sqcap \neg(\neg y \sqcap \bot) \overset{\text{P. 7(4a)}}{=} x \sqcap \neg\bot \\
&\overset{\text{T. 3(3a)}}{=} x \sqcap (\top \sqcap \top) \overset{\text{T. 3(1a)}}{=} x \sqcap \top \overset{\text{P.5 }(1a)}{=} x \sqcap x
\end{aligned}
$$

Towards associativity

Proposition 9

$(1a)\ x \sqcap \neg(x \sqcap y) = x \sqcap \neg y$ $\quad (1b)\ x \sqcup \lrcorner(x \sqcup y) = x \sqcup \lrcorner y$

$(2a)\ x \sqcap \neg(x \sqcap \neg y) = x \sqcap y$ $\quad (2b)\ x \sqcup \lrcorner(\lrcorner y \sqcup x) = x \sqcup y$

$(3a)\ x \sqcap \neg(\neg x \sqcap y) = x \sqcap x$ $\quad (3b)\ \lrcorner((x \sqcup x) \sqcup \lrcorner(\neg x \sqcup y)) = \lrcorner x$

$(4a)\ \neg x \sqcap \neg(x \sqcap y)) = \neg x$ $\quad (4b)\ \lrcorner x \sqcup \lrcorner(x \sqcup y)) = \lrcorner x$

$(5a)\ \neg x \sqcap \neg((x \sqcap y) \sqcap z) = \neg x$ $\quad (5b)\ \lrcorner x \sqcup \lrcorner((x \sqcup y) \sqcup z) = \lrcorner x$

$(6a)\ x \sqcap (y \sqcap \neg x) = \bot$ $\quad (6b)\ x \sqcup (y \sqcup \lrcorner x) = \top$

Proposition 10

$(1a)\ ((x \sqcap y) \sqcap z) \sqcap \neg x = \bot$ $\quad (1b)\ ((x \sqcup y) \sqcup z) \sqcup \lrcorner x = \top$

$(2a)\ \neg(x \sqcap y) \sqcap \neg(x \sqcap \neg y) = \neg x$ $\quad (2b)\ \lrcorner(x \sqcup y) \sqcup \lrcorner(\lrcorner y \sqcup x) = \lrcorner x$

$(3a)\ \neg(\neg(x \sqcap (y \sqcap z)) \sqcap z) \sqcap z = x \sqcap (y \sqcap z)$ $\quad (3b)\ \lrcorner(\lrcorner(x \sqcup (y \sqcup z)) \sqcup z) \sqcup z =$

Elimination of Wille's 3a., 3b

Proposition 11

$(1a)\ \neg(x \sqcap \neg(y \sqcap z)) = \neg(x \sqcap \neg y) \sqcap \neg(x \sqcap \neg z)$
$(1b)\ \lrcorner(x \sqcup \lrcorner(y \sqcup z)) = \lrcorner(x \sqcup \lrcorner y) \sqcup \lrcorner(x \sqcup \lrcorner z)$
$(2a)\ x \sqcap (\neg(x \sqcap \neg y) \sqcap \neg(x \sqcap \neg z)) = (y \sqcap x) \sqcap z$
$(2b)\ x \sqcup (\lrcorner(x \sqcup \lrcorner y) \sqcup \lrcorner(x \sqcup \lrcorner z)) = (y \sqcup x) \sqcup z.$

Theorem 12

$(a)\ x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z \quad (b)\ x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z.$

Elimination of Wille's 3a., 3b

Proposition 11

$(1a)\ \neg(x \sqcap \neg(y \sqcap z)) = \neg(x \sqcap \neg y) \sqcap \neg(x \sqcap \neg z)$
$(1b)\ \lrcorner(x \sqcup \lrcorner(y \sqcup z)) = \lrcorner(x \sqcup \lrcorner y) \sqcup \lrcorner(x \sqcup \lrcorner z)$
$(2a)\ x \sqcap (\neg(x \sqcap \neg y) \sqcap \neg(x \sqcap \neg z)) = (y \sqcap x) \sqcap z$
$(2b)\ x \sqcup (\lrcorner(x \sqcup \lrcorner y) \sqcup \lrcorner(x \sqcup \lrcorner z)) = (y \sqcup x) \sqcup z.$

Theorem 12

$(a)\ x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z \quad (b)\ x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z.$

$$
\begin{aligned}
x \sqcap (y \sqcap z) &\overset{\text{D1 (1a)}}{=} x \sqcap (z \sqcap y) \overset{\text{P10 (3a)}}{=} \neg(\neg(x \sqcap (z \sqcap y)) \sqcap y) \sqcap y \\
&\overset{\text{D1 (1a)}}{=} \neg(y \sqcap \neg(x \sqcap (z \sqcap y))) \sqcap y \\
&\overset{\text{P11 (1a)}}{=} (\neg(y \sqcap \neg x) \sqcap \neg(y \sqcap \neg(z \sqcap y))) \sqcap y \\
&\overset{\text{P9 (1a)}}{=} (\neg(y \sqcap \neg x) \sqcap \neg(y \sqcap \neg z)) \sqcap y \\
&\overset{\text{D1 (1a)}}{=} y \sqcap (\neg(y \sqcap \neg x) \sqcap \neg(y \sqcap \neg z)) \\
&\overset{\text{P11 (2a)}}{=} (x \sqcap y) \sqcap z
\end{aligned}
$$

Minimality: D-core's 5a., 5b.

$\sqcap$	a	b
a	a	a
b	a	b

$\sqcup$	a	b
a	a	b
b	b	b

$\wedge$	a	b
a	b	b
b	b	b

$\vee$	a	b
a	a	a
b	a	a

x	$\neg x$	$\lrcorner x$
a	a	b
b	a	b

- $\neg\neg(b \sqcap b) = \neg\neg b = \neg a = a \neq b = b \sqcap b$, failing (5a).
- $\lrcorner\lrcorner(a \sqcup a) = b \neq a = a \sqcup a$, failing (5b).
- $\bot = a$ and $\top = b$.
- $(\{a, b\}; \sqcap, \sqcup, \neg, \lrcorner, \top, \bot) \models (1a) - (4a), (1b) - (4b), (6a), (6b)$ and (7)

Bibliography

P. Howlader, L. Kwuida, C.-J. Liau, and M. Behrisch.
Towards a simplified theory of double boolean algebras: Axioms and representations.
Preprint arXiv:2601.01418v1, 2026.

R. Wille.
Boolean concept logic.
In B. Ganter and G. W. Mineau, editors, *Conceptual Structures: Logical, Linguistic, and Computational Issues*, pages 317–331, Berlin, Heidelberg, 2000. Springer Berlin Heidelberg.

Thank You