On modular lattices with complementation

Jan Kühr

Joint work with Václav Cenker, Ivan Chajda and Helmut Länger

Palacký University in Olomouc
Czechia

A lattice with complementation $\mathfrak{L} = (L, \vee, \wedge, ', 0, 1)$ is a complemented lattice with a fixed complementation $x \mapsto x'$.

A lattice with complementation $\mathfrak{L} = (L, \vee, \wedge, ', 0, 1)$ is a complemented lattice with a fixed complementation $x \mapsto x'$.

$\mathcal{M}$ = the class of modular LwCs

$\mathcal{M}_{DM}$ = the class of modular LwCs satisfying De Morgan's laws

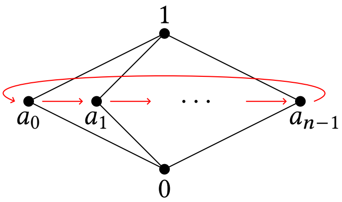
$$(x \vee y)' = x' \wedge y' \quad \text{and} \quad (x \wedge y)' = x' \vee y'$$

A lattice with complementation $\mathfrak{L} = (L, \vee, \wedge, ', 0, 1)$ is a complemented lattice with a fixed complementation $x \mapsto x'$.

$\mathscr{M}$ = the class of modular LwCs
$\mathscr{M}_{DM}$ = the class of modular LwCs satisfying De Morgan's laws

$$(x \vee y)' = x' \wedge y' \quad \text{and} \quad (x \wedge y)' = x' \vee y'$$

Example: $\mathfrak{M}_n = (M_n, \vee, \wedge, ', 0, 1)$ for $3 \leq n < \omega$



The variety $\mathscr{M}$ of MLwCs is an arithmetical and regular variety.

The variety $\mathscr{M}$ of MLwCs is an arithmetical and regular variety.

Consider the terms

$$m(x, y, z) = (x \wedge y \wedge z) \vee \big(x \wedge (x \wedge y)'\big) \vee \big(z \wedge (z \wedge y)'\big)$$

and

$$t_1(x, y, z) = (x \oplus y) \vee z \quad \text{and} \quad t_2(x, y, z) = \big((x \oplus y) \wedge z\big)' \wedge z,$$

where $\oplus$ is any 'symmetric difference' for $\mathscr{M}$, i.e., any binary term such that, in $\mathscr{M}$,

$$x \oplus x = 0, \quad x \oplus 0 = x, \quad x \oplus 1 = x', \quad \text{and} \quad x \oplus y = 0 \implies x = y;$$

The variety $\mathcal{M}$ of MLwCs is an arithmetical and regular variety.

Consider the terms

$$m(x, y, z) = (x \wedge y \wedge z) \vee \big(x \wedge (x \wedge y)'\big) \vee \big(z \wedge (z \wedge y)'\big)$$

and

$$t_1(x, y, z) = (x \oplus y) \vee z \quad \text{and} \quad t_2(x, y, z) = \big((x \oplus y) \wedge z\big)' \wedge z,$$

where $\oplus$ is any 'symmetric difference' for $\mathcal{M}$, i.e., any binary term such that, in $\mathcal{M}$,

$$x \oplus x = 0, \quad x \oplus 0 = x, \quad x \oplus 1 = x', \quad \text{and} \quad x \oplus y = 0 \implies x = y;$$

for instance, $x \oplus y$ could be

$$(x \vee y) \wedge (x \wedge y)' \quad \text{or} \quad (x \vee y) \wedge (x' \vee y'),$$

but not

$$(x \wedge y') \vee (x' \wedge y).$$

For every $\mathfrak{L} \in \mathscr{M}_{DM}$,

- complementation is injective;
- $\operatorname{Con} \mathfrak{L} = \operatorname{Con} L$.

For every $\mathfrak{L} \in \mathscr{M}_{DM}$,

- complementation is injective;
- $\operatorname{Con} \mathfrak{L} = \operatorname{Con} L$.

The variety $\mathscr{M}_{DM}$ is ideal determined; the terms

$$t_1(x, y_1, y_2) = x \wedge (y_1 \vee y_2) \quad \text{and} \quad t_2(x_1, x_2, y) = x_1 \wedge (x_2 \vee y) \wedge (x_1 \wedge x_2)'$$

form a basis of ideal terms for $\mathscr{M}_{DM}$.

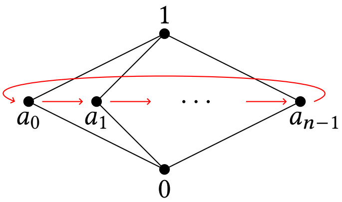
There are $2^{\aleph_0}$ subvarieties of $\mathscr{M}_{DM}$.

There are $2^{\aleph_0}$ subvarieties of $\mathscr{M}_{DM}$.

For every $\emptyset \neq S \subseteq \{3, 4, 5, \dots\}$, let $\mathscr{V}_S = \mathrm{V}(\mathfrak{M}_s \mid s \in S)$. The map $S \mapsto \mathscr{V}_S$ is injective:

Suppose $\mathfrak{M}_n \in \mathscr{V}_S$ but $n \notin S$, for some $n \geq 3$. Then $\mathfrak{M}_n \in \mathrm{HSP_U}(\mathfrak{M}_s \mid s \in S)$, i.e., $\mathfrak{M}_n \in \mathrm{H}(\mathfrak{K})$ where $\mathfrak{K} \in \mathrm{S}(\mathfrak{L})$ where $\mathfrak{L} \in \mathrm{P_U}(\mathfrak{M}_s \mid s \in S)$. Neither $\mathfrak{K}$ nor $\mathfrak{L}$ is a boolean algebra. Since the M_s are lattices of length 2 and width ≥ 3, so is L. Hence $L \cong M_\lambda$ for some $\lambda \geq 3$. Since $\mathfrak{K}$ is not a boolean algebra, $K \cong M_\kappa$ for some $\kappa \geq 3$. Then $\mathfrak{K}$ is a simple algebra, so $\mathfrak{M}_n \in \mathrm{H}(\mathfrak{K})$ must be isomorphic to $\mathfrak{K} \in \mathrm{S}(\mathfrak{L})$. However, since $n \notin S$, none of the $\mathfrak{M}_s$ has a subalgebra isomorphic to $\mathfrak{M}_n$, and hence neither does $\mathfrak{L}$; a contradiction.

Axiomatization of the variety $V(\mathfrak{M}_n)$?



For every integer $n \geq 3$, we consider the term

$$\tau_n(x) = \bigwedge_{2 \leq k < n} (x \vee x^{(k)}),$$

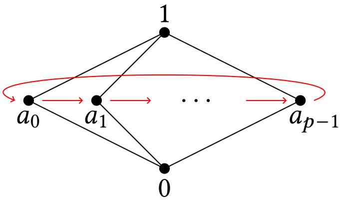
where $x^{(k)}$ is a shorthand for $x^{\prime\prime\cdots\prime}$ with k occurrences of $'$. Thus,

$$\tau_3(x) = x \vee x'', \quad \tau_4(x) = (x \vee x'') \wedge (x \vee x'''), \quad \text{etc.}$$

In $\mathfrak{M}_p$ with $p \geq n$ we have

$$\tau_n(a) = \bigwedge_{2 \leq k < n} \left(a \vee a^{(k)}\right) = \begin{cases} 0 & \text{for } a = 0, \\ 1 & \text{otherwise.} \end{cases}$$

This is not true in $\mathfrak{M}_p$ with $3 \leq p < n$, as $a_i^{(p)} = a_i$ and so $\tau_n(a_i) = a_i$ for every a_i.



For every integer $n \geq 3$, $\mathrm{V}(\mathfrak{M}_n)$ is axiomatized, relative to $\mathscr{M}_{DM}$, by the equations

$$x_0 \wedge \bigwedge_{1 \leq i < j \leq n} (x_i \vee x_j) \leq \bigvee_{1 \leq i \leq n} (x_0 \wedge x_i) \tag{M_n}$$

and

$$x \wedge (\tau_n(y) \vee z) = (x \wedge \tau_n(y)) \vee (x \wedge z). \tag{T_n}$$

For every integer $n \geq 3$, $\mathrm{V}(\mathfrak{M}_n)$ is axiomatized, relative to $\mathcal{M}_{DM}$, by the equations

$$x_0 \wedge \bigwedge_{1 \leq i < j \leq n} (x_i \vee x_j) \leq \bigvee_{1 \leq i \leq n} (x_0 \wedge x_i) \tag{M_n}$$

and

$$x \wedge (\tau_n(y) \vee z) = (x \wedge \tau_n(y)) \vee (x \wedge z). \tag{T_n}$$

Suppose $\mathfrak{L} \in \mathcal{M}_{DM}$ satisfies (M_n) and (T_n), is subdirectly irreducible, and is not a boolean algebra. Then $\mathrm{Neutr}\, L = \{0, 1\}$ because $\mathrm{Con}\, \mathfrak{L} = \mathrm{Con}\, L$.
By (T_n), the element $\tau_n(a) = \bigwedge_{2 \leq k < n} \left(a \vee a^{(k)}\right)$ is neutral for every a, and so $\tau_n(a) = 1$ for every $a \neq 0$. Thus $a \vee a' = a \vee a'' = \cdots = a \vee a^{(n-1)} = 1$ for every $a \neq 0$.
[...]
Then $L \cong M_\kappa$ for some $\kappa \geq n$, and (T_n) yields $\kappa \leq n$. Hence $\mathfrak{L} \cong \mathfrak{M}_n$.

In particular, $\mathrm{V}(\mathfrak{M}_3)$ is axiomatized, relative to $\mathcal{M}_{DM}$, by

$$x_0 \wedge (x_1 \vee x_2) \wedge (x_1 \vee x_3) \wedge (x_2 \vee x_3) \leq (x_0 \wedge x_1) \vee (x_0 \wedge x_2) \vee (x_0 \wedge x_3), \tag{M$_3$}$$

$$x \wedge (y \vee y'' \vee z) = (x \wedge (y \vee y'')) \vee (x \wedge z). \tag{T$_3$}$$

In particular, $\mathrm{V}(\mathfrak{M}_3)$ is axiomatized, relative to $\mathcal{M}_{DM}$, by

$$x_0 \wedge (x_1 \vee x_2) \wedge (x_1 \vee x_3) \wedge (x_2 \vee x_3) \leq (x_0 \wedge x_1) \vee (x_0 \wedge x_2) \vee (x_0 \wedge x_3), \tag{M$_3$}$$

$$x \wedge (y \vee y'' \vee z) = (x \wedge (y \vee y'')) \vee (x \wedge z). \tag{T$_3$}$$

In fact, $\mathrm{V}(\mathfrak{M}_3)$ is axiomatized, relative to $\mathcal{M}_{DM}$, by

$$x_0 \wedge (x_1 \vee x_2) \wedge (x_1 \vee x_3) \wedge (x_2 \vee x_3) \leq (x_0 \wedge x_1) \vee (x_0 \wedge x_2) \vee (x_0 \wedge x_3). \tag{M$_3$}$$

In particular, $\mathrm{V}(\mathfrak{M}_3)$ is axiomatized, relative to $\mathcal{M}_{DM}$, by

$$x_0 \wedge (x_1 \vee x_2) \wedge (x_1 \vee x_3) \wedge (x_2 \vee x_3) \leq (x_0 \wedge x_1) \vee (x_0 \wedge x_2) \vee (x_0 \wedge x_3), \tag{M$_3$}$$

$$x \wedge (y \vee y'' \vee z) = (x \wedge (y \vee y'')) \vee (x \wedge z). \tag{T$_3$}$$

In fact, $\mathrm{V}(\mathfrak{M}_3)$ is axiomatized, relative to $\mathcal{M}_{DM}$, by

$$x_0 \wedge (x_1 \vee x_2) \wedge (x_1 \vee x_3) \wedge (x_2 \vee x_3) \leq (x_0 \wedge x_1) \vee (x_0 \wedge x_2) \vee (x_0 \wedge x_3). \tag{M$_3$}$$

If $\mathfrak{L} \in \mathcal{M}_{DM}$ satisfies (M$_3$), then

- for every $a \in L$, $a \in \mathrm{Neutr}\, L$ iff $a = a''$;
- $\mathfrak{L}$ satisfies the equation $x \vee x'' = (x \vee x'')''$.

For every $n \geq 3$, the variety $\mathrm{V}(\mathfrak{M}_n)$ is a discriminator variety; the term

$$d(x, y, z) = \big(((x \oplus y) \vee (x \oplus y)'') \wedge x\big) \vee \big(((x \oplus y) \vee (x \oplus y)'')' \wedge z\big)$$

is a discriminator term for the algebra $\mathfrak{M}_n$.

Here $\oplus$ is any ‘symmetric difference’ for $\mathcal{M}_{DM}$, e.g.,

$$x \oplus y = (x \vee y) \wedge (x \wedge y)',$$

only for $n = 3$ we may take

$$x \oplus y = (x \wedge y') \vee (x' \wedge y).$$

For any $k \geq 1$ and $n \geq 3$, the free k-generator algebra in $\mathrm{V}(\mathfrak{M}_n)$ is

$$\mathfrak{M}_n{}^q \times \mathfrak{B}_2{}^r$$

where $q = \frac{(n+2)^k - 2^k}{n}$ and $r = 2^k$.

The exponents are

$$\frac{\text{number of valuations of } x_1, \ldots, x_k}{\text{number of automorphisms}}$$

THANK YOU!