Universal inverse monoids with the F-inverse property

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E-unitary and F-inverse monoids

- S – inverse semigroup; $E(S)$ – semilattice of idempotents of S.
- σ - minimum group congruence on S: $a \,\sigma\, b$ if and only if there is $e \in E(S)$ such that $ae = be$.
- S is called E-unitary if σ is idempotent pure, that is, satisfies

$$a \,\sigma\, e \text{ for some } e \in E(S) \;\Rightarrow\; a \in E(S).$$

- If each σ-class contains a maximum element, S is called F-inverse.
- Each F-inverse semigroup is a monoid.
- Each F-inverse monoid is E-unitary but the converse is not true.
- Example: the free X-generated inverse monoid is F-inverse.

F-inverse monoids in enriched signature

- S – F-inverse monoid.
- $a^{\mathfrak{m}} := \max\{b \in S \colon b \;\sigma\; a\}$, for each $a \in S$.
- $a \mapsto a^{\mathfrak{m}}$ is a unary operation on S.
- We add the operation ${}^{\mathfrak{m}}$ to the type $(\cdot, {}^{-1}, 1)$ of S and get the algebra $(S; \cdot, {}^{-1}, {}^{\mathfrak{m}}, 1)$ of signature $(2, 1, 1, 0)$.
- Proposition. An algebra $(S; \cdot, {}^{-1}, {}^{\mathfrak{m}}, 1)$ is an F-inverse monoid if and only if $(S; \cdot, {}^{-1}, 1)$ is an inverse monoid and:

  1) $a^{\mathfrak{m}} \geq a$ for all $a \in S$,

  2) $a^{\mathfrak{m}} = (ae)^{\mathfrak{m}}$ for all $a \in S$ and $e \in E(S)$.
- Corollary. F-inverse monoids form a variety of algebras (Kynion, 2018).

$M(G,X)$

- G – an X-generated group
- $M(G,X)$ – all pairs (Γ, g) where Γ is a finite connected subgraph of $\mathrm{Cay}(G,X)$ containing the vertices 1 and g. It is an inverse monoid with the operations

$$(\Delta, g)(\Xi, h) = (\Delta \cup g\Xi, gh)$$

and

$$(\Delta, g)^{-1} = (g^{-1}\Delta, g^{-1}).$$

- $M(G,X)$ is an X-generated inverse monoid with respect to $x \mapsto (\Gamma_x, [x])$, $x \in X$. It is the universal X-generated E-unitary inverse monoid with maximum group image G (Margolis, Meakin, 1989).
- Defining relations for $M(G,X)$: $w^2 = w$ for all $w \in \overline{X}^*$ such that $w = 1$ holds in G.
- When $G = FG(X)$ then $w = 1$ in G if and only if $red(w)$ is the empty word (such words are called Dyck words). It follows that $FIM(X) \cong M(FG(X), X)$.

$F(G, X)$

- $F(G, X)$ – all pairs (Γ, g) where Γ is a finite (not necessarily connected) subgraph of $\mathrm{Cay}(G, X)$ containing the vertices 1 and g
- $F(G, X)$ is an X-generated F-inverse monoid; operations are defined similarly as on $M(G, X)$ and additionally

$$(\Delta, g)^{\mathfrak{m}} = (\{1, g\}, g),$$

 where $\{1, g\}$ is the graph with no edges and two vertices $1, g$. It is the universal X-generated F-inverse monoid with maximum group image G (Auinger, GK, Szendrei, 2021).
- Defining relations for $F(G, X)$: $w^{\mathfrak{m}} = w$ for all terms w such that $w = 1$ holds in G.
- When $G = FG(X)$ then the free F-inverse monoid $FFM(X)$ is isomorphic to $F(FG(X), X)$.
- In both $M(G, X)$ and $F(F, X)$ we have $(\Delta, g) \leq (\Xi, h)$ if and only if $\Xi \subseteq \Delta$ and $g = h$, and $(\Delta, g)\ \sigma\ (\Xi, h)$ if and only if $g = h$.

The universal F-inverse monoid $M_F(G, X)$

- For every σ-class of $M(G, X)$ we identify any two maximal elements of this σ-class.
- Denote the quotient by $M_F(G, X)$.
- $M_F(G, X)$ is an F-inverse monoid and is the universal F-inverse quotient of $M(G, X)$ such that the quotient map preserves maximal elements of σ-classes.
- $\overline{X} = X \cup X^{-1}$.
- $G = \mathrm{Gp}\langle X \mid w_i = 1 \ (i \in I)\rangle$ where $w_i \in \overline{X}^*$ are cyclically reduced words.
- $w \in \overline{X}^*$ is cyclic with respect to (G, X) if w labels a simple cycle in the Cayley graph $\mathrm{Cay}(G, X)$.

Presentation for $M_F(G,X)$

- θ_G – the set of all pairs of words (u,v), where $u,v \in \overline{X}^*$ are non-empty, such that uv is cyclic with respect to (G,X).
- $\theta_G^\sharp$ – the congruence on $M(G,X)$ generated by the set

$$\{((\Gamma_u,[u]),(\Gamma_{v^{-1}},[v^{-1}])) : (u,v) \in \theta_G\}.$$

Theorem 1. $M_F(G,X) = M(G,X)/\theta_G^\sharp$. Therefore, $M_F(G,X)$ is presented, as an X-generated inverse monoid, by the relations:

(i) $w^2 = w$ for all $w \in \overline{X}^*$ such that $w = 1$ holds in G,

(ii) $u = v^{-1}$ for all $u,v \in \overline{X}^*$ such that $(u,v) \in \theta_G$.

Sketch of the proof of Theorem 1

- The maximal elements in the $\sigma/\theta_G^\sharp$-class corresponding to $g \in G$ are of the form $(\Pi, g)\theta_G^\sharp$ where Π is a simple path in $\mathrm{Cay}(G, X)$ connecting the vertices 1 and g
- we need to show that (Π_1, g) and (Π_2, g) must be $\theta_G^\sharp$-related for all simple paths Π_1 and Π_2 from 1 to g we must have that .
- The cycle $\Pi_1\Pi_2^{-1}$ does not need to be simple. We carefully analyse how Π_1 and Π_2 interweave:

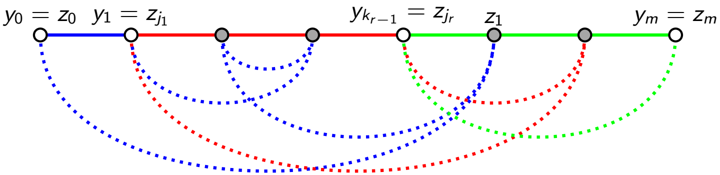

Figure: The full line represents the path Π_1 and the dotted line represents the path Π_2.

Model for $M_F(G, X)$

- A subgraph $\Delta \subseteq \text{Cay}(G, X)$ will be called cyclic if for every its edge e it contains all edges of any simple cycle containing e.
- Δ° be the smallest cyclic subgraph of $\text{Cay}(G, X)$ containing Δ.
- The map $\Delta \mapsto \Delta^\circ$ is a G-invariant, finitary closure operator on the family of all connected subgraphs of $\text{Cay}(G, X)$.
- The subgraphs of the form $\Delta = F^\circ$ for a finite connected subgraph $F \subseteq \text{Cay}(G, X)$ are called compact.
- $S_\circ$ – the set of all pairs (Δ, g) where $g \in G$ and Δ is a compact connected subgraph of $\text{Cay}(G, X)$ containing 1 and g. It is an X-generated inverse monoid (follows from Szakács, 2024).
- The generators of $S_\circ$ are the pairs $(\Gamma_x^\circ, [x])$, $x \in X$.

Proposition Let G be an X-generated group. Then $M_F(G, X) \cong S_\circ$.

Simplifying the presentation of $M_F(G, X)$

- G has the property $(\dagger)$ if it admits a presentation $G = \mathrm{Gp}\langle X \mid w_i = 1\ (i \in I)\rangle$ such that all the w_i are cyclic.
- All one-relator groups have this property (Weinbaum, 1972), and this is witnessed by any of their presentations by one defining relation involving a cyclically reduced word.

Theorem 2 Let G be an X-generated group with the property $(\dagger)$ and let

$$G = \mathrm{Gp}\langle X \mid w_i = 1\ (i \in I)\rangle$$

be a of its presentations witnessing this property. Then the inverse monoid $M_F(G, X)$ is presented by the relations

$$u = v^{-1}$$

for all non-empty words $u, v \in \overline{X}^*$ such that uv is a cyclic conjugate of w_i for some $i \in I$.

Van Kampen diagrams

- Let $G = \mathrm{Gp}\langle X \mid w_i = 1\ (i \in I)\rangle$ where all words w_i are cyclically reduced. A van Kampen diagram D over G is a cell 2-complex along with its embedding into $\mathbb{R}^2$ such that the following properties hold:
  - there is a vertex (a 0-cell) called the base vertex;
  - the 1-skeleton of D is a directed $\overline{X}$-labelled graph closed under $^{-1}$.
  - D is connected and simply connected (i.e., if every cycle in the underlying graph is 'filled' with 2-cells);
  - for every region (2-cell) R of D, every vertex in the cycle formed by its boundary ∂R and every choice of the two possible directions, the word labeling the cycle based at the chosen vertex is a cyclic conjugate of either w_i of w_i^{-1} for some $i \in I$.
- The area of D is the number of its regions.
- The boundary word $w \in \overline{X}^*$ of D is the label of the path that can be read clockwise around the boundary ∂D (start at the base vertex).
- w is a boundary word for $D \Leftrightarrow D$ is a van Kampen diagram for w.
- van Kampen Lemma Let $w \in \overline{X}^*$ be a reduced word. Then $w = 1$ holds in G if and only if w has a van Kampen diagram over G.

(Part of) proof of Theorem 2.

- (Part of proof) Let $w = 1$ holds in G. We show that $w^2 = w$ follows from the relations in the formulation.
- Induction on the area of the van Kampen diagram D for w.
- If the area is 0 then the diagram in question is in fact a tree and so the word read at the boundary ∂D is a Dyck word. So $w^2 = w$.
- Assume that the area of D is positive. Pick any region R, containing at least one edge e from ∂D. Let $x \in \overline{X}$ be its label.
- Let u be the word labelling the remainder of the boundary of R. Then $x = u^{-1}$ is one of the relations from the formulation.
- Now delete the edge e from D. We get a new diagram D' of smaller area whose boundary word w' is obtained from w by replacing the occurrence of the letter x by u^{-1}. Hence, $w = w'$ is a consequence of the relations given in the formulation.
- By IH: $(w')^2 = w'$ follows from the given relations. Thus $w^2 = w$.

Special inverse monoids which are quotients of $M_F(G, X)$.

- Let $M = \mathrm{Inv}\langle X \mid w_i = 1 \ (i \in I)\rangle$.
- The word w is linked with respect to M if every pair of its adjacent letters is matching with respect to M.
- Let $G = \mathrm{Gp}\langle X \mid w_i = 1 \ (i \in I)\rangle$ be a presentation of the group G with the property that each of its relator words w_i is cyclic.
- M satisfies all the relations from Theorem 2 if and only if each word w_i, $i \in I$, is linked with respect to M.
- A one-relator special inverse monoid $M = \mathrm{Inv}\langle X \mid w = 1\rangle$, where w is cyclically reduced, is a quotient of $M_F(G, X)$ if and only if w is linked with respect to M.
- Let $w \in \overline{X}^*$ be a cyclically reduced word and $M = \mathrm{Inv}\langle X \mid w = 1\rangle$. The special inverse monoid M is a quotient of $M_F(X)$ if and only if every piece w_i in the decomposition $w = w_1 \cdots w_k$ of w into minimal invertible pieces has length at most two.

F-inverse quotients of $M(G, X)$

- To obtain a maximal F-inverse canonical quotient of $M(G, X)$ whose maximal group quotient is G we must impose the following relations, for every $g \in G$:
  - Let $g \in G$, $g \neq 1$, and let Π_i, $i \in I$ be all simple paths from 1 to g in $\mathrm{Cay}(G, X)$. Then (Π_i, g) are the maximal elements of $M(G, X)$ over g. Pick one of these elements (Π, g) and impose relations $(\Pi_i, g) \leq (\Pi, g)$ for all $i \in I$.
  - This way we obtain a maximal F-inverse quotient of $M(G, X)$.
  - $M_F(G, X)$ is a quotient of any of these F-inverse monoids.
  - Can these quotients be different from $M_F(G, X)$?

Example

- Take $X = G \setminus \{1\}$. Let $g \in G$. Pick Π_g to be the path from 1 to g with one edge, $(1, g, g)$. Impose the relations $(\Pi_g, g) \geq (\Delta, g)$ where Δ is a any other path from 1 to g.
- (Π_g, g) is the maximum element of its σ-class in the quotient.
- The associated closure operator $\Delta \mapsto \Delta^c$ takes a finite subgraph Δ of $\mathrm{Cay}(G, X)$ and adds to it all the missing edges of $\mathrm{Cay}(G, X)$ between pairs of vertices of Δ. That is, Δ^c is an induced subgraph of $\mathrm{Cay}(G, X)$ and is thus determined by its vertices $V(\Delta^c) = V(\Delta)$ only (which is a finite subset of G).
- It now easily follows that the quotient F inverse monoid is isomorphic to the Birget-Rhodes expansion $BR(G)$ of G.
- Since, for distinct non-identical $g, h \in G$ we have $g = h \cdot h^{-1}g$, it follows that $hh^{-1} \geq gg^{-1}$ whence $gg^{-1} = hh^{-1}$. Since $gg^{-1} \neq hh^{-1}$ in $BR(G)$, $M_F(G, X)$ is a proper quotient of $BR(G)$.
- Work in progress. Describe all one-relator special X-generated F-inverse monoids.