Topologizing Endomorphism Monoids

Joint with S. Bardyla

$\mathcal{T}$ - Semigroup topology on S

- $*: (S, \mathcal{T}) \times (S, \mathcal{T}) \to (S, \mathcal{T})$
  is continuous

"A guide to topological reconstruction on endomorphism monoids and polymorphism clones"
Marimon, Pinsker

Pointwise topology

$$S \subseteq X^X$$

$$\{f \in S \mid f(x) = y\} : x, y \in X$$

$$(f_n)_{n \in \mathbb{N}} \longrightarrow f \iff \forall x \in X, f_n(x) \text{ is eventually } f(x).$$

Small topologies

Zariski $\{x \in S \mid s_0 x s_1 \ldots x s_n \neq t_0 x t_1 \ldots x t_m\}$

$s_0, \ldots, s_n, t_1, \ldots, t_m \in S^1$.

Proposition (E, Jonušas, Mesyan, Mitchell, Morayne, Péresse)

If X is a set and $S \leq X^X$ such that

- S contains the constant maps
- for all $x \in X$, there is $f_x \in S$ with finite image with $f_x^{-1}(x) = \{x\}$

Then the Zariski topology is the pointwise topology

Proposition (E, Jonušas, Mitchell, Péresse, Pinsker)

If $\mathbb{A}$ is a countable ω-categorical homogeneous arsfacere relational structure with no algebraicity, then Zariski is pointwise for all S with

$$\mathrm{Emb}(\mathbb{A}) \leq S \leq \mathrm{End}(\mathbb{A})$$

Theorem (Pinsker, Schindler)
There is an ω-categorical structure G where the Zariski and Pointwise topogies on $\mathrm{End}(G)$ differ.

Theorem (Pinsker, Schindler)
If $\mathbb{A}$ is ω-categorical, has no algebaicity, has a mobile core and either

i) the model-complete core is finite or
ii) the model-complete core is infinite with no algebaicity

then Zariski on $\mathrm{End}(\mathbb{A})$ is pointwise.

Large topologies

Polish = complete metric + 2^{nd} countable

Semigroups with a finest Polish topology:

$\mathbb{N}^{\mathbb{N}}$, $I_{\mathbb{N}}$, $P_{\mathbb{N}}$, $\mathrm{Inj}(\mathbb{N})$, $C(2^{\mathbb{N}})$, $C([0,1]^{\mathbb{N}})$

– E, Jonušas, Mesyan, Mitchell, Morayne, Péresse

End(X) where X is the random graph, random digraph, random poset, $\omega \mathbb{K}_n$, $n \mathbb{K}_\omega$.

– E, Jonušas, Mitchell, Péresse, Pinsker

$\mathrm{End}(\mathbb{Q}, \leq)$ – Pinsker and Schindler

$\mathrm{End}(\mathbb{N}, \leq)$

- trivial group of units
- Zariski topdogy is pointwise topology
- infinitely many Polish topology
- has a finest Polish topology

This topology is defined using

$$\mathrm{End}^{\infty}(\mathbb{N}, \leq) \leq \mathrm{End}(\mathbb{N}, \leq)$$

$\mathrm{End}^{\infty}(\mathbb{N}, \leq)$

- $\mathcal{J}$-simple
- Zariski is pointwise
- has unique Polish topology.

Finest Polish topology for $\mathrm{End}(\mathbb{N}, \leq)$:

$\mathrm{End}^{\infty}(\mathbb{N}, \leq)$ - pointwise

disjoint union

$\mathrm{End}^{<\infty}(\mathbb{N}, \leq)$ - discrete

$\mathrm{End}(\mathbb{Z}, \leq)$

- group of units is $\mathbb{Z}$
- infinitely many Polish topologies
- has a finest Polish topology

$$\{ f \in \mathrm{End}(\mathbb{Z}, \leq) \mid f(x) = y \} \quad : \quad x, y \in \mathbb{Z}$$

$$\{ f \in \mathrm{End}(\mathbb{Z}, \leq) \mid \min(\mathrm{im}(f)) = x \}, \{ f \in \mathrm{End}(\mathbb{Z}, \leq) \mid \max(\mathrm{im}(f)) = x \} : x \in \mathbb{Z}$$

$$\{ f \in \mathrm{End}(\mathbb{Z}, \leq) \mid f \text{ is unbounded above} \}, \{ f \in \mathrm{End}(\mathbb{Z}, \leq) \mid f \text{ is unbounded below} \}$$

Partial Symmetries

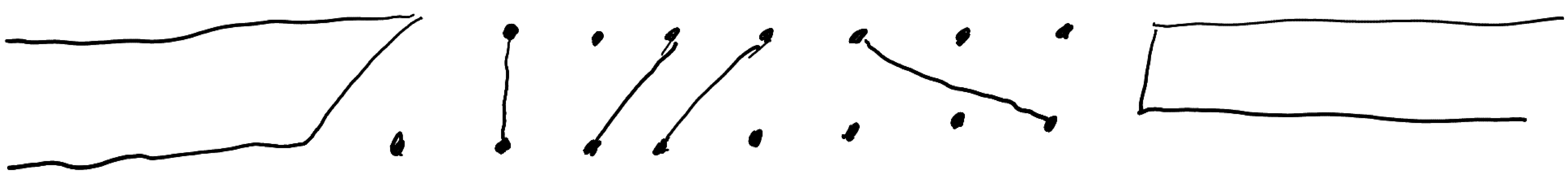
we have a new operation $x \to x^{-1}$ which we want to be continuous and use in Zariski sets.

PartialIso$(\mathbb{N}, \leq)$

- inverse Zariski topology is $\mathcal{I}_4$
- unique Polish inverse semigroup topology is $\mathcal{I}_4$
- finest Polish inverse semigroup topology is $\mathcal{I}_4$
- infinitely many Polish semigroup topologies

PartialIso($\mathbb{Z}, \leq$)

- inverse Zariski topology is $\mathcal{T}_4$
- non-unique Polish inverse semigroup topology!?
- infinitely many Polish semigroup topologies
- has a finest Polish inverse semigroup topology.



$\mathrm{End}(\mathbb{N}, <)$

- no maximal 2^{nd} countable
- has $2^{\aleph_0}$ Polish semigroup topologies
- also $\geq 2^{\aleph_0}$ non-Polish 2^{nd} countable metrizable topologies

fun question to ponder

1) Recall that (in ZFC) each uncountable Polish space has cardinality $2^{\aleph_0}$.

2) If we give any Polish space the operation $(a,b) \mapsto b$, then it becomes a Polish semigroup

Thus there is a semigroup with

$2^{2^{\aleph_0}}$ Polish topologies.

in ZFC

Semigroup	# Polish semigroup topologies
$\mathbb{R}^{\mathbb{R}}$	0
$\mathbb{N}^{\mathbb{N}}$	1
$I_{\mathbb{N}}$	$\aleph_0$
$\mathrm{End}(\mathbb{N}, <)$	$2^{\aleph_0}$
$(\mathbb{R}, (a,b) \mapsto b)$	$2^{2^{\aleph_0}}$

is this all possibilities?