Operators on complemented lattices

Ivan Chajda and Helmut Länger

Palacký
University
Olomouc

Introduction

Let $\mathrm{L} = (L, \vee, \wedge, 0, 1)$ be a bounded lattice and $a \in L$. An element b of L is called a *complement* of a if $a \vee b = 1$ and $a \wedge b = 0$. The lattice L is called *complemented* if any of its elements has a complement.

Often lattices with an additional unary operation, usually denoted by $'$, are studied where for each $a \in L$ the element a' denotes its complement. In such a case this unary operation is called a *complementation*. However, in complemented lattices we do not assume the complement being unique. This is the case with our present paper.

It is worth noticing that in a distributive complemented lattice the complement is unique. However, this need not be the case in modular complemented lattices. For example, consider the lattice $\mathrm{M}_n = (M_n, \vee, \wedge, 0, 1)$ (for $n > 1$) depicted in Figure 1:

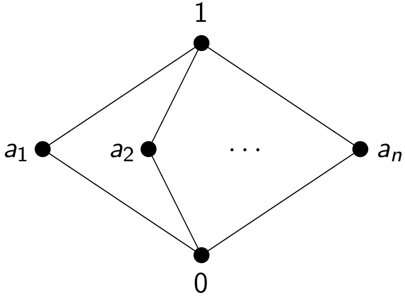

Fig. 1

The lattice M_n

Then for every $i, j \in \{1, \ldots, n\}$ with $i \neq j$, the element a_j is a complement of a_i.

Sometimes, for lattices with complementation, we ask if this complementation is *antitone*, i.e. if $x \leq y$ implies $y' \leq x'$, or if it is an *involution*, i.e. $x'' = x$. In distributive complemented lattices the complementation turns out to be unique, antitone and an involution. In such a case the lattice is a Boolean algebra.

Within modular lattices the situation may be different. Consider the complemented modular lattice $\mathsf{L} = (L, \vee, \wedge, 0, 1)$ visualized in Figure 2:

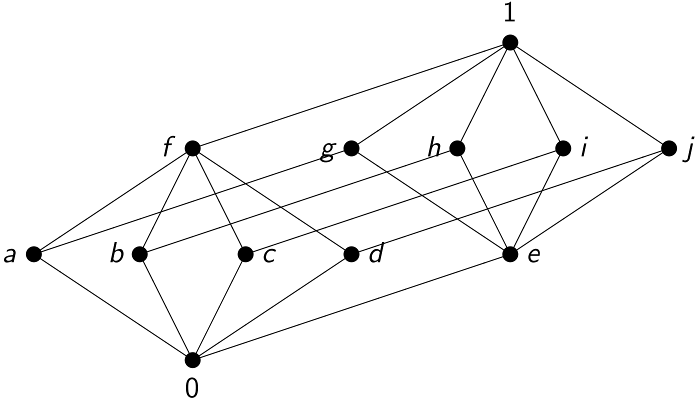

Fig. 2

Complemented modular lattice

Evidently; L is a complemented lattice. We have several choices for defining a complementation $'$. If we define $'$ by

x	0	a	b	c	d	e	f	g	h	i	j	1
x'	1	h	i	j	g	f	e	b	c	d	a	0

then it is not an involution. If we define $'$ by

x	0	a	b	c	d	e	f	g	h	i	j	1
x'	1	h	i	j	g	f	e	d	a	b	c	0

then it is an antitone involution and hence $\mathrm{L} = (L, \vee, \wedge, ', 0, 1)$ is a so-called *orthomodular lattice* (see e.g. [1] for the definition).

Hence, not every modular lattice endowed with a complementation must be orthomodular. Of course, not every orthomodular lattice is modular (see [1]).
If $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ is a complemented lattice in which the complementation is not introduced in form of a unary operation then we need not distinguish between the complements of a given element a of L. Hence we will work with the whole set of complements of a. Within this paper we will use this approach.

We start by introducing some lattice-theoretical concepts.
All complemented lattices considered within this paper are assumed to be non-trivial, i.e. to have a bottom element 0 and a top element 1 with $0 \neq 1$.
Let $(L, \vee, \wedge, 0, 1)$ be a complemented lattice and $A, B \subseteq L$. We define:

$$
\begin{aligned}
A \vee B &:= \{x \vee y \mid x \in A \text{ and } y \in B\}, \\
A \wedge B &:= \{x \wedge y \mid x \in A \text{ and } y \in B\}, \\
A \leq B &\text{ if } x \leq y \text{ for all } x \in A \text{ and all } y \in B, \\
A \leq_1 B &\text{ if for every } x \in A \text{ there exists some } y \in B \text{ with } x \leq y, \\
A \leq_2 B &\text{ if for every } y \in B \text{ there exists some } x \in A \text{ with } x \leq y.
\end{aligned}
$$

The operator $^+$

Let $\mathsf{L} = (L, \vee, \wedge, 0, 1)$ be a complemented lattice. For $a \in L$ we define

$$a^+ := \{x \in L \mid a \vee x = 1 \text{ and } a \wedge x = 0\},$$

i.e. a^+ is the set of all complements of a. Since L is complemented, we have $a^+ \neq \emptyset$ for all $a \in L$. For every subset A of L we put

$$A^+ := \{x \in L \mid a \vee x = 1 \text{ and } a \wedge x = 0 \text{ for all } a \in A\}.$$

Observe that A^+ may be empty, e.g. $L^+ = \emptyset$ (and $\emptyset^+ = L$). In the following we often identify singletons with their unique element.

Example

For the lattice N_5 depicted in Figure 3:

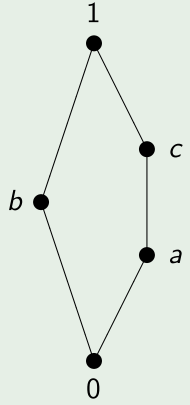

Fig. 3

Non-modular lattice N_5

we have

x	0	a	b	c	1
x^{+}	1	b	ac	b	0
x^{++}	0	ac	b	ac	1

Here and in the following within tables we sometimes write *abc* instead of $\{a, b, c\}$. For the lattice M_3 visualized in Figure 4

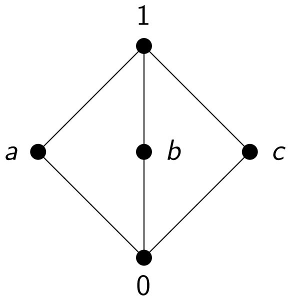

Fig. 4

Modular lattice M_3

we have

x	0	a	b	c	1
x^+	1	bc	ac	ab	0
x^{++}	0	a	b	c	1

Let us note that M_3 satisfies the identity $x^{++} \approx x$.

Example

For the example from Figure 2 we have

x	0	a	b	c	d	e	f	g	h	i	j	1
x^+	1	hij	gij	ghj	ghi	f	e	bcd	acd	abd	abc	0
x^{++}	0	a	b	c	d	e	f	g	h	i	j	1

Recall the concept of a *Galois connection* which is often used in lattices. The pair $(^+, ^+)$ is the Galois connection between $(2^L, \subseteq)$ and $(2^L, \subseteq)$ induced by the relation

$$\{(x, y) \in L^2 \mid x \vee y = 1 \text{ and } x \wedge y = 0\}.$$

From this we conclude

$$\begin{gathered} A \subseteq A^{++}, \\ A \subseteq B \Rightarrow B^+ \subseteq A^+, \\ A^{+++} = A^+, \\ A \subseteq B^+ \Leftrightarrow B \subseteq A^+ \end{gathered}$$

for all $A, B \subseteq L$. Since $A \subseteq A^{++}$ we have that $A^{++} \neq \emptyset$ whenever $A \neq \emptyset$. A subset A of L is called *closed* if $A^{++} = A$. Let $\mathrm{Cl(L)}$ denote the set of all closed subsets of L. Then clearly $\mathrm{Cl(L)} = \{A^+ \mid A \subseteq L\}$. Because of $A^+ \cap A^{++} = \emptyset$ for all $A \subseteq L$ we have that $\big(\mathrm{Cl(L)}, \subseteq, ^+, \emptyset, L\big)$ forms a complete ortholattice with

$$\bigvee_{i \in I} A_i = \left(\bigcup_{i \in I} A_i \right)^{++},$$
$$\bigwedge_{i \in I} A_i = \bigcap_{i \in I} A_i$$

for all families $(A_i; i \in I)$ of closed subsets of L.
Next we describe the basic properties of the operator $^+$.

Proposition

Let $\mathsf{L} = (L, \vee, \wedge, 0, 1)$ *be a complemented lattice and* $a \in L$. *Then the following hold:*

(i) $a \in a^{++}$ *and* $a^{+++} = a^{+}$,

(ii) $(x^{+}, \leq)$ *is an antichain for every* $x \in L$ *if and only if* L *does not contain a sublattice isomorphic to* N_5 *containing* 0 *and* 1,

(iii) $(a^{+}, \leq)$ *is convex,*

(iv) *if the mapping* $x \mapsto x^{++}$ *from* L *to* 2^{L} *is not injective then* L *does not satisfy the identity* $x^{++} \approx x$.

In the lattice N_5 from Example 9 the mapping $x \mapsto x^{++}$ is not injective since $a \neq c$ and $a^{++} = c^{++}$. According to Proposition 3 (iv), this lattice does not satisfy the identity $x^{++} \approx x$, e.g. $a^{++} = \{a, c\} \neq a$.

Corollary

Let $(L, \vee, \wedge, 0, 1)$ be a complemented modular lattice, $a \in L$ and A a non-empty subset of L. According to Proposition 3 (iii), $(a^+, \leq)$ is an antichain. Let $b \in A$. Then $A^+ \subseteq b^+$ and hence also $(A^+, \leq)$ is an antichain. Since a^+ is a non-empty subset of L we finally conclude that $(a^{++}, \leq)$ is an antichain, too.

In case of finite L we can even prove the following.

Proposition

Let $(L, \vee, \wedge, 0, 1)$ be a finite complemented lattice such that $x \mapsto x^{++}$ is injective and $a \in L$ and assume $a^{++} \neq a$. Then there exists some $b \in a^{++}$ with $b^{++} = b$.

The relationship between the operator $^{+}$ and the partial order relation of L is illuminated in the following result.

Proposition

Let $(L, \vee, \wedge, 0, 1)$ be a complemented lattice and consider the following statements:

(i) $x^{+} \vee y^{+} \leq_1 (x \wedge y)^{+}$ *for all* $x, y \in L$,

(ii) *for all* $x, y \in L$, $x \leq y$ *implies* $y^{+} \leq_1 x^{+}$,

(iii) $(x \vee y)^{+} \leq_1 x^{+} \wedge y^{+}$ *for all* $x, y \in L$.

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii).

Our next task is to characterize the property that a complemented lattice $\mathbf{L}$ satisfies the identity $x^{++} \approx x$. From Example 9 we know that if $\mathbf{L}$ is not modular then this identity need not hold. Hence we restrict ourselves to complemented modular lattices.

Theorem

Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a complemented modular lattice. Then the following are equivalent:

- *(i)* $\mathbf{L}$ *satisfies the identity* $x^{++} \approx x$,
- *(ii)* *for every* $x \in L$ *and each* $y \in x^{++}$ *there exists some* $z \in y^{+}$ *satisfying either* $(x \vee y) \wedge z = 0$ *or* $(x \wedge y) \vee z = 1$.

The operator $\to$

Let $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ be an orthomodular lattice. Recall that the operation ϕ_x defined by $\phi_x(y) := x \wedge (x' \vee y)$ for all $x, y \in L$ was introduced by U. Sasaki in [5] and [6] and is called the *Sasaki projection* (see e.g. [1]) or *Sasaki hook* alias *Sasaki operation*, see [3]; its dual, i.e. the operation ψ_x defined by $\psi_x(y) := x' \vee (x \wedge y)$ for all $x, y \in L$ is then called the *dual Sasaki projection*. It was shown by the authors in [3] that if we use these Sasaki operations in order to define

$$x \to y := x' \vee (x \wedge y),$$
$$x \cdot y := (x \vee y') \wedge y$$

for all x, y belonging to the base set of the orthomodular lattice $\mathbf{L}$ then the operations $\to$ and $\cdot$ form an adjoint pair, i.e.

$$x \cdot y \leq z \text{ if and only if } x \leq y \to z$$

for all $x, y, z \in L$.

This motivated us to introduce our next operators in a similar way where, however, instead of the element x' we use the set x^+. Hence, for a complemented lattice $(L, \vee, \wedge, 0, 1)$, $a, b \in L$ and $A, B \subseteq L$ we define

$$a \to b := a^+ \vee (a \wedge b),$$
$$A \to B := A^+ \vee (A \wedge B).$$

Observe that $A \to B = \emptyset$ whenever $A^+ = \emptyset$.

Example

For the lattice from Figure 2 we have e.g.

$$\begin{aligned}
a \to b &= \{h, i, j\} \vee (a \wedge b) = \{h, i, j\} \vee 0 = \{h, i, j\} = a^+,\\
a \to f &= \{h, i, j\} \vee (a \wedge f) = \{h, i, j\} \vee a = 1,\\
a \to g &= \{h, i, j\} \vee (a \wedge g) = \{h, i, j\} \vee a = 1,\\
a \to h &= \{h, i, j\} \vee (a \wedge h) = \{h, i, j\} \vee 0 = \{h, i, j\} = a^+,\\
f \to e &= e \vee (f \wedge e) = e \vee 0 = e,\\
g \to h &= \{b, c, d\} \vee (g \wedge h) = \{b, c, d\} \vee e = \{h, i, j\} = a^+.
\end{aligned}$$

In the following we study the relationship between $\rightarrow$ and $\wedge$.

Theorem

Let $(L, \vee, \wedge, 0, 1)$ be a complemented modular lattice and $a, b, c \in L$. Then the following hold:

(i) *If $a \leq_1 b \rightarrow c$ then $a \wedge b \leq c$,*

(ii) *$a \wedge b \leq c$ if and only if $a \wedge b \leq_1 b \rightarrow c$.*

For complemented lattices, the operator $\rightarrow$ satisfies a lot of properties common in residuated structures.

Theorem

Let $(L, \vee, \wedge, 0, 1)$ be a complemented lattice and $a, b, c \in L$. Then the following hold:

(i) $a \rightarrow 0 = a^+$ *and* $1 \rightarrow a = a$,

(ii) *If* $a \leq b$ *then* $a \rightarrow b = 1$,

(iii) $a \rightarrow b = 1$ *if and only if* $a \wedge b \in a^{++}$,

(iv) *if* $b \in a^+$ *then* $a \rightarrow b = a^+$,

(v) *if* $b \leq c$ *then* $a \rightarrow b \leq_i a \rightarrow c$ *for* $i = 1, 2$,

(vi) *if* $a \rightarrow b = a \rightarrow c = 1$ *and* a^{++} *is closed with respect to* $\wedge$ *then* $a \rightarrow (b \wedge c) = 1$,

(vii) *if* $a^{++} \subseteq b^{++}$ *and* $a \rightarrow b = 1$ *then* $b \rightarrow a = 1$.

Let us note that the converse of Theorem 10 (ii) does not hold in general. For example, consider the lattice N_5 from Example 9. Then $c \to a = c^+ \vee (c \wedge a) = b \vee a = 1$, contrary to the fact that $c > a$. However, if x is a minimal element of x^{++}, then we can prove the following.

Proposition

Let $(L, \vee, \wedge, 0, 1)$ be a complemented lattice and $a \in L$. Then the following are equivalent:

(i) *For all $x \in L$, $a \to x = 1$ is equivalent to $a \leq x$,*

(ii) *a is a minimal element of a^{++}.*

We are going to show how the operator $\to$ is related to the connective implication in a propositional calculus.

Theorem

Let $\mathrm{L} = (L, \vee, \wedge, 0, 1)$ *be a complemented modular lattice and* $a, b \in L$. *Then the following hold:*

(i) $a \wedge (a \to b) = a \wedge b \leq b$ (Modus Ponens),

(ii) *if* $a^+ \leq b^+$ *then* $(a \to b) \wedge b^+ = a^+$ (Modus Tollens),

(iii) *if* $c \in a \to b$ *then* $a \to c = a \to b$,

(iv) $a \to (a \to b) = a \to b$,

(v) *if* $a^+ \leq b$ *then* $a \to b = b$,

Proposition

Let $n > 1$ *and* $a, b, c \in M_n$. *Then*

$$a \wedge b \leq c \text{ if and only if } a \leq_1 b \to c.$$

The operator $\odot$

Similarly as it was done in Section 3 concerning the operator $\rightarrow$, also here we define the new operator $\odot$ by means of the generalized Sasaki projection.
For a complemented lattice $(L, \vee, \wedge, 0, 1)$, $a, b \in L$ and $A, B \subseteq L$ we define

$$a \odot b := b \wedge (a \vee b^{+}),$$
$$A \odot B := B \wedge (A \vee B^{+}).$$

It is evident that $\odot$ need neither be commutative nor associative, but it is idempotent, i.e. it satisfies the identity $x \odot x \approx x$ (cf. Proposition 14 (iii)).
We list some basic properties of the operator $\odot$.

Proposition

Let $\mathsf{L} = (L, \vee, \wedge, 0, 1)$ *a complemented lattice and* $a, b, c \in L$*. Then the following hold:*

(i) $0 \odot a = a \odot 0 = 0$,

(ii) $1 \odot a = a \odot 1 = a$,

(iii) $a \wedge b \leq a \odot b \leq b$ *and if* $b \leq a$ *then* $a \odot b = b$,

(iv) *if* $a \leq b$ *then* $a \odot c \leq_i b \odot c$ *for* $i = 1, 2$,

(v) *if* L *is modular then* $a \leq b$ *if and only if* $a \odot b = a$ *and, moreover,* $(a \odot b) \odot b = a \odot b$.

Example

The "operation tables" for $\odot$ for the lattices N_5 and M_3 (see Example 9) are as follows:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	c	a
b	0	0	b	0	b
c	0	a	0	c	c
1	0	a	b	c	1

N_5

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	$0b$	$0c$	a
b	0	$0a$	b	$0c$	b
c	0	$0a$	$0b$	c	c
1	0	a	b	c	1

M_3

Contrary to the relatively week relationship between $\to$ and $\wedge$, for $\odot$ and $\to$ we can prove here a kind of adjointness.

Theorem

Let $(L, \vee, \wedge, 0, 1)$ be a complemented modular lattice and $a, b, c \in L$. Then

$$a \odot b \leq c \text{ if and only if } a \leq b \to c.$$

Deductive systems

Deductive systems are often introduced in algebras forming an algebraic formalization of a non-classical propositional calculus. These are subsets of the algebra in question containing the logical constant 1 and representing the derivation rule Modus Ponens. Since our operator $\to$ shares a number of properties with the non-classical logical connective implication, we define this concept also for complemented lattices.

Definition

A *deductive system* of a complemented lattice $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ is a subset D of L satisfying the following conditions:

$$1 \in D,$$

$$\text{if } a \in D,\ b \in L \text{ and } a \to b \subseteq D \text{ then } b \in D.$$

Since the intersection of deductive systems of L is again a deductive system of L, the set of all deductive systems of L forms a complete lattice $\operatorname{Ded}\mathrm{L}$ with respect to inclusion with bottom element $\{1\}$ and top element L.
The relationship between deductive systems and filters is described in the following results.

Lemma

Let $\mathrm{L} = (L, \vee, \wedge, 0, 1)$ *be a complemented lattice and* D *a deductive system of* L. *Then the following hold:*

(i) D *is an order filter of* L,

(ii) *if* $x \to y \subseteq D$ *for all* $x, y \in D$ *then* D *is a filter of* L.

If L is, moreover, modular then we can prove also the following.

Proposition

Let $\mathsf{L} = (L, \vee, \wedge, 0, 1)$ *be a complemented modular lattice and* F *a filter of* L*. Then* F *is a deductive system of* L*.*

In the remaining part of this section we investigate when a given deductive system D may induce an equivalence relation Φ such that $D = [1]\Phi$, i.e. D being its kernel. We start with the following definition.

Definition

For every complemented lattice $(L, \vee, \wedge, 0, 1)$ and every deductive system D of L put

$$\Theta(D) := \{(x, y) \in L^2 \mid x \to y, y \to x \subseteq D\}.$$

From Theorem 10 (ii) we get that $\Theta(D)$ is reflexive and, by definition, it is symmetric.
It is easy to see that every congruence on a complemented modular lattice induces a deductive system.

Proposition

Let $(L, \vee, \wedge, 0, 1)$ *be a complemented modular lattice and* $\Phi \in \mathrm{Con}(L, \wedge)$. *Then the following hold:*

(i) $[1]\Phi$ *is a deductive system of* L*,*

(ii) $\Theta([1]\Phi) \subseteq \Phi$*.*

The previous proposition shows that we need a certain compatibility of the induced relation $\Theta(D)$ with the lattice operations in order to show D to be the kernel of $\Theta(D)$. For this sake, we define the following properties.

Definition

Let $(L, \vee, \wedge, 0, 1)$ be a complemented lattice and Φ an equivalence relation on L. We say that Φ has the *Substitution Property with respect to* $^+$ if

$$(a, b) \in \Phi \text{ implies } a^+ \times b^+ \subseteq \Phi,$$

and the *Substitution Property with respect to* $\rightarrow$ if

$$(a, b) \in \Phi \text{ implies } (a \rightarrow c) \times (b \rightarrow c) \subseteq \Phi \text{ for all } c \in L.$$

Such an equivalence relation Φ can be related with the equivalence relation induced by its kernel $[1]\Phi$ and, moreover, this kernel is a deductive system.

Theorem

Let $\mathsf{L} = (L, \vee, \wedge, 0, 1)$ *be a complemented lattice and* Φ *an equivalence relation on* L *having the Substitution Property with respect to* $\to$. *Then the following hold:*

(i) Φ *has the Substitution Property with respect to* $^+$,

(ii) $[1]\Phi$ *is a deductive system of* L,

(iii) $\Phi \subseteq \Theta([1]\Phi)$.

Now we are able to relate deductive systems with equivalence relations induced by them provided these deductive systems satisfy a certain compatibility condition defined as follows.

Definition

Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a complemented lattice and D a deductive system of $\mathbf{L}$. We call D a *compatible deductive system* of $\mathbf{L}$ if it satisfies the following two additional conditions for all $a, b, c, d \in L$:

If $a \to b \subseteq D$ and $x \to (c \to d) \subseteq D$ for all $x \in a \to b$ then $c \to d \subseteq D$,

if $a \to b, b \to a \subseteq D$ then $x \to (b \to c) \subseteq D$ for all $x \in a \to c$.

Now we show that also conversely as in previous Theorem, a compatible deductive system induces an equivalence relation having the Substitution Property with respect to $\rightarrow$.

Theorem

Let $\mathsf{L} = (L, \vee, \wedge, 0, 1)$ *be a complemented lattice and* D *a compatible deductive system of* L*. Then the following hold:*

(i) $\Theta(D)$ *is an equivalence relation on* L *having the Substitution Property with respect to* $\rightarrow$*,*

(ii) $[1]\big(\Theta(D)\big) = D$*.*

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The end!

Thanks for your attention!!