Graphs on algebras

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Finally, I observe that many of the easier arguments immediately extend to arbitrary algebras. In some cases the results are the same as for groups. More commonly, they raise questions which may involve considering special kinds of algebra in order to make progress.
I hope you will consider trying your hand at some of the questions, either in general or in a variety you are most familiar with.

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The prototype example of what I am talking about was introduced by Brauer and Fowler in 1955 (although they did not call it that). Given a group G, we define a graph $\Gamma(G)$ with vertex set G in which x and y are joined if and only if $xy = yx$. This is the commuting graph of G.

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They used this graph to prove that, given a group H with a central involution z, there are only finitely many finite simple groups G containing an involution whose centraliser is isomorphic to H.

The commuting graph of a group

This was arguably the first step on the long journey to the Classification of Finite Simple Groups (CFSG). Many subsequent papers took a particular group H and worked out which simple groups have H as an involution centraliser. One person who struck gold was Dieter Held: the groups $L_5(2)$, M_{24} and the Held group have isomorphic involution centralisers.

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- it is defined on the group G, purely in terms of the group operation;
- so it is invariant under the automorphism group of G.

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We have to restrict ourselves to interesting graphs or relational structures. Exactly which ones to choose is a matter of taste.

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For now, let me restrict the algebraic structures to groups, and the relational structures to graphs. Let us approach the question a different way:

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- What is the complexity of deciding whether a graph is the commuting graph of a group?

Some answers

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Landau proved that, given a positive integer k, there are only finitely many finite groups having just k conjugacy classes.

Most subsequent research concentrated on the question of finding good bounds for the order of a group with k conjugacy classes. But here is a different direction. We define a graph whose vertices are the conjugacy classes of the group. Landau bounds the group order by a function of the number of vertices. Can we bound it by a function of some graph parameter?

An extension of Landau's theorem

The solvable conjugacy class graph of a finite group G is the graph whose vertices are the conjugacy classes of G, classes C_1 and C_2 joined if there exist $g_i \in C_i$ (for $i = 1, 2$) such that the group $\langle g_1, g_2 \rangle$ is solvable.

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This was proved by Parthajit Bhowal, Rajat Kanti Nath, Benjamin Sambale and me. We used CFSG in the proof, but only in a "light-touch" way, and we conjecture that its use can be avoided. Also, we do not have any bounds!

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The last of these is empty if G is not 2-generated; but we know that all finite simple groups are 2-generated, so it is particularly useful for these.

Defining classes of groups

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These classes can be defined by forbidden induced subgraphs: for example, cographs forbid 4-vertex paths, while chordal graphs forbid induced cycles of length greater than 3.
So we can ask: for which groups G is $\Gamma(G)$ perfect, or a cograph, or chordal (these classes are subgroup-closed)? Also, for which groups G is $\Gamma_1(G) = \Gamma_2(G)$, for two of these graph types?

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The first holds because the directed power graph (with $x \to y$ if $y \in \langle x \rangle$, and a loop at each vertex) is a partial preorder (i.e. reflexive and transitive); it can be extended to a partial order by putting a total order on each indifference class.

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The second is a recent result of Bubboloni, Fumagalli and Praeger (arXiv `2510.18073`). More on this (and the third) later.
See also the talk by Samir Zahirović coming up next.

Beautiful graphs

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Take the smallest Mathieu simple group M_{11}, with order 7920. Form the difference of the power graph and enhanced power graph (whose edges are those in the latter but not the former). Perform twin reduction (identify two vertices if they have the same open or closed neighbourhood) until no such pairs remain.

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The resulting graph is semiregular bipartite, with blocks of sizes 165 and 220, and valencies 4 and 3 for the two blocks; it has diameter 10, girth 10 (surprisingly large), and automorphism group M_{11}.

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Note that for a graph with n vertices, the input size is polynomial in n; we take advantage of the fact that groups of order n have short descriptions (of size polynomial in $\log n$).

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- *G is cyclic.*

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Second, the theory of groups is well-developed with a very wide range of models. Indeed, no part of algebra apart from linear algebra has a comparable theory, and vector spaces are not so interesting from this point of view. (Some researchers have looked at graphs on vector spaces, but usually they include a basis as part of the structure.)

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I will make a few remarks on more general definitions.

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For any algebra A, we have the notion of the subalgebra $\langle S\rangle$ generated by a subset S (the intersection of all subalgebras containing S). So if the definition of a graph only involves subalgebras, it will work in any algebra. As usual we write $\langle\{x\}\rangle$ as $\langle x\rangle$.

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Note that any algebra A has a unique mimimal subalgebra $E(A)$, the intersection of all the subalgebras; it can be defined as the subalgebras generated by the constants (the values of the 0-ary operations).

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The names may be inappropriate in general algebras!

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Suppose that the enhanced power graph Γ of an algebra A is a cograph, but fails to be chordal.
It cannot contain an induced cycle of length greater than 4 (since this contains a 4-vertex path). Assume that (a, b, c, d) is a 4-cycle, Then there exists z such that $a, b \in Z = \langle z \rangle$. Choose z such that Z is maximal. Then $z \not\sim c$; for if $z \sim c$, then by maximality $c \in Z$, and $a \sim c$, which is not so. Similarly $z \not\sim d$.

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A "dual" argument handles the intersection power graph.

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Since the subgroups of a cyclic group are totally ordered if and only if it has prime power order, we obtain the following. An EPPO group is a group in which every element has prime power order. The EPPO groups were classified by Rolf Brandl in 1981, following earlier work by Graham Higman and Michio Suzuki.

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Corollary

The power graph and enhanced power graph of a group G coincide if and only if G is an EPPO group.

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- Two elements x and y are joined in the generating graph if $\langle x, y \rangle = A$.

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There are links with earlier graphs, to which we now turn. (These were noted for groups, but they hold in general.)

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What, if anything, can be said in general?

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These graphs could be defined for arbitrary algebras for which we have a subalgebra-closed class to take the place of $\mathcal{C}$. The choice is likely to be dependent on the type of algebra considered.

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The cases of equality here are highly non-trivial. They depend on knowing that the minimal non-nilpotent or non-solvable groups are 2-generated; these depend on the classification of minimal non-nilpotent groups by Schmidt, and Thompson's classification of N-groups.

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In general, where such classifications don't exist, are there similar results, perhaps involving the independence or rank graphs?

More detail on some of this is in the surveys

- Peter J. Cameron, Graphs defined on groups, *Internat. J. Group Theory* **11** (2022), 53–107.
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