Dualities for bilattices and their applications

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Bilattices, which provide an algebraic tool for simultaneously modelling knowledge and truth, were introduced by N. D. Belnap in a 1977 paper entitled *How a computer should think*. Prioritised default bilattices include not only Belnap's four values, for 'true' ($\boldsymbol{t}$), 'false'($\boldsymbol{f}$), 'contradiction' ($\top$) and 'no information' ($\perp$), but also indexed families of default values for simultaneously modelling degrees of knowledge and truth.

In our paper [3] we introduced a new class $\{\, \mathbf{J}_n \mid n \in \omega \,\}$ of default bilattices for use in prioritised default logic. The first of these bilattices, $\mathbf{J}_0$, is Belnap's famous four-element bilattice known as $\mathcal{FOUR}$ [1], while for $n \geqslant 1$, the bilattice $\mathbf{J}_n$ provides a new algebraic structure for dealing with inconsistent and incomplete information. The importance of our prioritised default bilattices in comparison with those previously studied is that in our family there is no distinction between the level at which the contradictions or agreements take place. Any contradictory response that includes some level of truth ($\boldsymbol{t}_i$) and some level of falsity ($\boldsymbol{f}_j$) is registered as a total contradiction ($\top$) and a total lack of consensus ($\perp$). This can lead to improvements in existing applications of default bilattices.

Prioritised default bilattices now have many applications in artificial intelligence. Sakama [8] studied default theories based on a 10-valued bilattice and applications to inductive logic programming. Shet, Harwood and Davis [9] proposed a prioritised multi-valued default logic for identity maintenance in visual surveillance. Encheva and Tumin [5] applied default logic based on a 10-element default bilattice in an intelligent tutoring system as a way of resolving problems with contradictory or incomplete input.

Bilattices are algebras $\mathbf{A} = \langle A; \otimes, \oplus, \wedge, \vee, \neg\rangle$ with two lattice structures, a knowledge lattice $\mathbf{A}_{\mathrm{k}} = \langle A; \otimes, \oplus\rangle$, with associated *knowledge order* $\leqslant_{\mathrm{k}}$, and a truth lattice $\mathbf{A}_{\mathrm{t}} = \langle A; \wedge, \vee\rangle$, with associated *truth order* $\leqslant_{\mathrm{t}}$, along with an involutive negation $\neg$ which is an order automorphism of $\mathbf{A}_{\mathrm{k}}$ and a dual order automorphism of $\mathbf{A}_{\mathrm{t}}$. Our algebra $\mathbf{J}_n$ is a *prioritised default bilattice* as it is equipped with two hierarchies of nullary operations, $\boldsymbol{t}_i$ and $\boldsymbol{f}_i$, that represent, respectively, true and false by default. We refer the reader to [3] for motivation and background on bilattices in general and prioritised default bilattices in particular.

In our approach we address mathematical rather than logical aspects of our prioritised default bilattices. The lack of the much-used product representation in our context led us to develop a concrete representation via the theory of natural dualities (in the sense of [2]). In [3], we presented a natural duality between the variety $\boldsymbol{\mathcal{V}}_n$ generated by $\mathbf{J}_n$ and a category $\boldsymbol{\mathfrak{X}}_n$ of multi-sorted topological structures. At the beginning of our talk we briefly recall this duality. Our first main aim in this talk is to describe the dual category $\boldsymbol{\mathfrak{X}}_n$. We begin by giving an axiomatisation of the multi-sorted category $\boldsymbol{\mathfrak{X}}_n$ and then describe an isomorphic category $\boldsymbol{\mathfrak{Y}}_n$ of single-sorted topological structures. The objects of $\boldsymbol{\mathfrak{Y}}_n$ are Priestley spaces endowed with a continuous retraction in which the order has a natural ranking. Then we describe the Priestley dual $\mathrm{H}(\mathbf{A}^\flat)$ of the underlying bounded distributive lattice $\mathbf{A}^\flat$ of an algebra $\mathbf{A}$ in $\boldsymbol{\mathcal{V}}_n$. As an application of we show that the size of the free algebra $\mathbf{F}_{\boldsymbol{\mathcal{V}}_n}(1)$ is given by a polynomial in n of degree 6. This result is presented below:

Theorem 1. *Let $n \in \omega\backslash\{0\}$. Then the cardinality of the 1-generated free algebra in the variety $\boldsymbol{\mathcal{V}}_n = \mathrm{Var}(\mathbf{J}_n)$ is*

$$|F_{\boldsymbol{\mathcal{V}}_n}(1)| = \tfrac{1}{2}\big(n^6 + 10n^5 + 42n^4 + 102n^3 + 157n^2 + 148n + 72\big).$$

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Except the result presenting the natural duality between the variety $\boldsymbol{\mathcal{V}}_n$ generated by $\mathbf{J}_n$ and the category $\boldsymbol{\mathfrak{X}}_n$ of multi-sorted topological structures from [3], the results presented here are based on our second paper [4] that was accepted for publication in May 2022.

References

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