Marginalia to [Sh:365]

Conference in honor of Saharon Shelah's 80th birthday
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Background

Theorem (Ramsey, 1930)

For every coloring $c : [\mathbb{N}]^2 \to 2$ *there exists an infinite* $A \subseteq \mathbb{N}$ *such that* c *takes just one color over all pairs of elements of* A*.*

Theorem (Sierpiński, 1933)

There exists a coloring $c : [\mathbb{R}]^2 \to 2$ *such that for every uncountable* $A \subseteq \mathbb{R}$*,* c *takes both colors over pairs of elements of* A*.*

Definition

$\kappa \nrightarrow [\lambda]^2_\theta$ asserts the existence of a coloring $c : [\kappa]^2 \to \theta$ such that for every $A \in [\kappa]^\lambda$, c takes all colors over pairs of elements of A.

Was Sierpiński right?

Sierpiński established that $2^{\aleph_0} \nrightarrow [\aleph_1]^2_2$. In particular, $\aleph_1 \nrightarrow [\aleph_1]^2_2$.
What about three colors? more?

Theorem (Todorčević, 1987)

$\aleph_1 \nrightarrow [\aleph_1]^2_{\aleph_1}$.

*The above is where Todorčević's method of *walks on ordinals* was introduced.

Theorem (Shelah, 1988)

$2^{\aleph_0} \rightarrow [\aleph_1]^2_3$ *is consistent modulo a large cardinal hypothesis.*

*The above is Saharon's first paper in his series named *Was Sierpiński right?*
See [Sh:276], [Sh:288], [Sh:481] and [Sh:546].

Our series

There is another series of papers that could have the same title (to be explained below), but as ‘Was Sierpiński right?’ is already owned by Shelah, me and my co-author, Tanmay Inamdar, went instead with a homage to it:

[U1] T.I. and A.R., *Was Ulam right? I: Basic theory and subnormal ideals*, Topology Appl., 323(C): 108287, 53pp, 2023.

[U2] T.I. and A.R., *Was Ulam right? II: Small width and general ideals*, Algebra Universalis, 85(2): 14, 47pp, 2024.

[U3] T.I. and A.R., *Was Ulam right? III: Indecomposable ideals*, submitted April 2025.

Simply put, the series’ goal is to find extensions of Solovay’s decomposition theorem. We do so by studying Sierpiński’s onto$(\cdots)$ principle and its variant unbounded$(\cdots)$.

Solovay's decomposition theorem

One of the earliest gems a set theory student will see:

Theorem (Solovay, 1971)

Every stationary subset of a regular uncountable cardinal κ may be decomposed into κ many stationary sets.

Roughly speaking, Todorčević's (first) proof of $\aleph_1 \nrightarrow [\aleph_1]^2_{\aleph_1}$ consists of three steps:

1. Find a map $t : [\omega_1]^2 \to [\omega_1]^{<\omega} \setminus \{\emptyset\}$ such that for every $A \in [\omega_1]^{\omega_1}$, the union $\bigcup\{t(\alpha,\beta) \mid (\alpha,\beta) \in [A]^2\}$ covers a club in ω_1;
2. Find a choice function $c : [\omega_1]^2 \to \omega_1$ with $c(\alpha,\beta) \in t(\alpha,\beta)$ for all $\alpha < \beta < \omega_1$ maintaining that $\{c(\alpha,\beta) \mid (\alpha,\beta) \in [A]^2\}$ covers a club in ω_1 for every $A \in [\omega_1]^{\omega_1}$;
3. Fix a partition $\langle S_i \mid i < \omega_1\rangle$ of ω_1 into stationary sets, and define $c^* : [\omega_1]^2 \to \omega_1$ via $c^*(\alpha,\beta) := i$ iff $c(\alpha,\beta) \in S_i$. By step 2, c^* witnesses $\aleph_1 \nrightarrow [\aleph_1]^2_{\aleph_1}$.

This talk

I have so much to say about generalizations of each of these three steps. There are countless relevant works by people in this room, as well by myself and my co-authors... But having 30 minutes, I won't even have the chance to say much about [U1] and [U2]! (though, you're invited to watch a previous talk: `https://youtu.be/xsVz4t8DB74`)

Now, let me tell you the main result of this talk.

Partition relations for inaccessibles

Theorem [Sh:365]

If κ is a weakly inaccessible admitting a stationary set that does not reflect at regulars, then $\kappa \nrightarrow [\kappa]^2_\theta$ for every $\theta < \kappa$.

It remained open whether it is possible to get $\theta = \kappa$ in the same scenario. Personally, I learned about this problem from Eisworth during his visit to Toronto in 2012.

> **1.2. Remark.** (1) Is this hard? A posteriori it does not look so, but we have worked hard on it several times without success (worse: produced several false proofs).

Quote from [Sh:572]

Partition relations for inaccessibles

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It remained open whether it is possible to get $\theta = \kappa$ in the same scenario. Personally, I learned about this problem from Eisworth during his visit to Toronto in 2012.

Theorem (with Inamdar [U3])

If κ is a weakly inaccessible admitting a stationary set that does not reflect at regulars, then $\kappa \nrightarrow [\kappa]^2_\kappa$.

The proof from [Sh:365]

Shelah's proof follows the prototypical three steps:

1. Use walks on ordinals to obtain a map $t : [\kappa]^2 \to [\kappa]^{<\omega} \setminus \{\emptyset\}$ such that for every $A \in [\kappa]^\kappa$, the union $\bigcup\{t(\alpha,\beta) \mid (\alpha,\beta) \in [A]^2\}$ is measure one with respect to some club-guessing ideal;
2. Find a choice function $c : [\kappa]^2 \to \kappa$ with $c(\alpha,\beta) \in t(\alpha,\beta)$ for all $\alpha < \beta < \kappa$ maintaining that $\{c(\alpha,\beta) \mid (\alpha,\beta) \in [A]^2\}$ is measure one for every $A \in [\kappa]^\kappa$;
3. For each $\theta < \kappa$, prove that κ may be partitioned into θ many positive sets with respect to said ideal. By composing c with such a partition, conclude $\kappa \nrightarrow [\kappa]^2_\theta$.

Our proof keeps steps 1 and 2, so only need to revisit step 3.

Step 3 of the proof from [Sh:365]

Claim 3.3 Suppose

(i) λ regular, $\sigma = \mathrm{cf}\sigma < \lambda$, $\tau < \lambda$

(ii) $S \subseteq \lambda$ stationary, $\bar{C} = \langle C_\delta : \delta \in S \rangle$ an S-club system

(iii) $\bar{I} = \langle I_\delta : \delta \in S \rangle$, I_δ a σ-complete ideal on C_δ[8].

(iv) $(\sup_\delta |C_\delta|)^+$ is $< \lambda$ or just
$\mathfrak{r} = \{\theta : \tau \leq \theta = \mathrm{cf}\ \theta < \lambda$, and every I_δ is θ-indecomposable$\}$
is unbounded in $\{\mathrm{cf}(\delta) : \delta < \lambda\}$

(v) $\mathrm{id}_p(\bar{C}, \bar{I})$ is a proper ideal on λ

(vi) $A^* \subseteq \lambda$, $A^* \notin \mathrm{id}_p(\bar{C}, \bar{I})$.

Then there is a partition of A^* to τ sets not in $\mathrm{id}_p(\bar{C}, \bar{I})$ provided that at least one of the following holds:

(α) λ is inaccessible not Mahlo

(β) λ is inaccessible and some stationary

$$S^0 \subseteq \{\delta : \delta < \lambda \text{ and } \mathrm{cf}\delta < \sigma \vee \mathrm{cf}(\delta) \in \mathfrak{r}\}$$

reflect in no inaccessible and
(a) $\delta \in S^0 \Rightarrow \mathrm{cf}(\delta) \in \mathfrak{r}$
or
(b) $\tau \leq \sigma$

(γ) $\lambda = \mu^+$, μ singular, $\mathrm{pp}(\mu) > \mu^+$ and $\mu > \sigma > \mathrm{cf}(\mu)$,

(δ) $\lambda = \mu^+$, μ singular, $\mathrm{pp}(\mu) > \mu^+$, $\sigma \leq \mathrm{cf}\mu$ and every σ-complete uniform filter on $\mathrm{cf}(\mu)$ is not weakly τ-saturated.

Proofs of Solovay's theorem

Solovay's original proof used generic ultrapowers, but let us start with a baby case of decomposing a given stationary subset S of $E^\kappa_\lambda := \{\beta < \kappa \mid \mathrm{cf}(\beta) = \lambda\}$.
For each $\beta \in S$, fix a club C_β in β of order-type λ.
Let $d : \lambda \times S \to \kappa$ be the map that satisfies that $d(\eta, \beta)$ is the η^{th}-element of C_β.

Key claim. *There is $\eta < \lambda$ such that for every $\epsilon < \kappa$, there is $\beta \in S$ with $d(\eta, \beta) \geq \epsilon$.*

By Fodor's lemma, it follows that for an η as in the claim,
the set $T := \{\tau < \kappa \mid \{\beta \in S \mid d(\eta, \beta) = \tau\}$ is stationary$\}$ is cofinal in κ.
Clearly, for all $\tau \neq \tau'$ in T, $\{\beta \in S \mid d(\eta, \beta) = \tau\}$ and $\{\beta \in S \mid d(\eta, \beta) = \tau'\}$ are two disjoint stationary sets. So we are done.

Proofs of Solovay's theorem (cont.)

In the general case, when given a stationary $S \subseteq \kappa$, first shrink S to a stationary $S' \subseteq S \cap \mathrm{acc}(\kappa)$ on which there is a sequence $\vec{C} = \langle C_\beta \mid \beta \in S' \rangle$ satisfying the following two:

1. $\vec{C}$ is a *C-sequence*, i.e., for each $\beta \in S'$, C_β is a club in β;
2. $\vec{C}$ is *amenable*, i.e., for every club $D \subseteq \kappa$, the set $\{\beta \in S' \mid D \cap \beta \subseteq C_\beta\}$ is nonstationary.

Fix a map $d : \kappa \times S' \to \kappa$ satisfying $d(\eta, \beta) = \min(C_\beta \setminus \eta + 1)$ provided $\eta < \beta$.

Application of amenability

There is an $\eta < \kappa$ such that for every $\epsilon < \kappa$, there is a $\beta \in S'$ with $d(\eta, \beta) \geq \epsilon$.

- The rest follows as in the previous slide.

How is it related to Ulam's work?

Theorem (Ulam, 1930)

Suppose that $\kappa = \lambda^+$ is a successor cardinal.
Then there exists a map $d : \lambda \times \kappa \to \kappa$ satisfying the following.
For every uniform κ-complete ideal I over κ, for every $B \in I^+$,
there is an $\eta < \lambda$ for which $\{\tau < \kappa \mid \{\beta \in B \mid d(\eta, \beta) = \tau\} \in I^+\}$ is cofinal in κ.

This applies to sets like $B := \{\alpha + 1 \mid \alpha < \kappa\}$, where the analog of Fodor's lemma fails and a formula like $d(\eta, \beta) := \min(C_\beta \setminus \eta + 1)$ makes little sense.

How is it related to Ulam's work?

Theorem (Ulam, 1930)

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For every uniform κ-complete ideal I over κ, for every $B \in I^+$,
there is an $\eta < \lambda$ for which $\{\tau < \kappa \mid \{\beta \in B \mid d(\eta, \beta) = \tau\} \in I^+\}$ is cofinal in κ.

Theorem (Hajnal, 1967)

Suppose κ is weakly inacc. admitting a stationary set that does not reflect at regulars.
Then there exists a map $\boxed{d : \kappa \times \kappa \to \kappa}$ satisfying the following.
For every uniform κ-complete ideal I over κ, for every $B \in I^+$,
there is an $\boxed{\eta < \kappa}$ for which $\{\tau < \kappa \mid \{\beta \in B \mid d(\eta, \beta) = \tau\} \in I^+\}$ is cofinal in κ.

By [U2], a map as in Hajnal's theorem exists iff κ admits a *nontrivial* C-sequence.

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Hajnal's theorem is irrelevant to [Sh:365], since club-guessing ideals are not κ-complete.

Club-guessing ideals

Club-guessing ideals over κ are not κ-complete, but they are θ-indecomposable for all but finitely many $\theta \in \mathrm{Reg}(\kappa)$. Our goal is to prove they are not weakly κ-saturated.

Recall

For an infinite regular cardinal θ, a uniform ideal I over κ is:

- θ-indecomposable iff it is closed under increasing unions of length exactly θ;
- weakly θ-saturated iff κ cannot be decomposed into θ-many I^+-sets.

The proof of [Sh:365, Claim 3.3]

Two proofs are provided in the book: one using generic ultrapowers, and one that is combinatorial and makes use of a least function.

Definition

For an ideal I over κ, a function $f : \kappa \to \kappa$ is

- bounded modulo I iff there is an $\epsilon < \kappa$ such that $\{\beta < \kappa \mid f(\beta) \leq \epsilon\} \in I^*$;
- least function modulo I iff f is not bounded modulo I, but every $g <_I f$ is.

*As far as we can tell, it was introduced in (Ketonen, 1973) as a ‘first function’.
Soon after, e.g. (Kanamori, 1976), the convention became ‘least function’.

The proof of [Sh:365, Claim 3.3] (cont.)

The combinatorial proof establishes (towards a contradiction) the existence of a least function (in fact, an eub). It then makes two modifications to the previous argument:

1. Replace the formula $d(\eta,\beta) := \min(C_\beta \setminus \eta+1)$ by $d(\eta,\beta) := \min(C_{f(\beta)} \setminus \eta+1)$ for f a least function modulo our I;
2. Given $B \in I^+$, instead of finding $\eta<\kappa$ and a cofinal set $T \subseteq \kappa$ such that $\{\beta \in B \mid d(\eta,\beta)=\tau\} \in I^+$ for every $\tau \in T$,
   find an $\eta<\kappa$ and a cofinal set $T \subseteq \kappa$ such that $\{\beta \in B \mid \tau_0 \leq d(\eta,\beta) < \tau_1\} \in I^+$ for every pair $\tau_0<\tau_1$ of ordinals from T.

Naming the last variation

For an ideal I over κ, unbounded$^{\square}(I,\kappa)$ asserts the existence of an upper-regressive colouring $d:[\kappa]^2 \rightarrow \kappa$ with the property that, for every $B \in I^+$, there are an $\eta<\kappa$ and a cofinal $T \subseteq \kappa$ such that $\{\beta \in B \mid \tau_0 \leq d(\eta,\beta) < \tau_1\} \in I^+$ for all $\tau_0<\tau_1$ from T.

For a family $\mathcal{I}$ of ideals, we write unbounded$^{\square}(\mathcal{I},\kappa)$ to assert the existence of a simultaneous witness for all $I \in \mathcal{I}$.

Our idea

Replace the formula $d(\eta, \beta) := \min(C_{f(\beta)} \setminus \eta + 1)$ by $d(\eta, \beta) := \min(C_{g(\eta,\beta)} \setminus \eta + 1)$ for some 2D function g arising from walks on ordinals.

In some scenarios, it suffices to let g be the simple function *last* isolated in our 2021 paper with Zhang. The naming of the function comes from the fact that it depends on the last step of the walk from β down to η.

Unlike least, a last function always exists (though its quality may vary).

Extension of Ulam's theorem

Theorem (with Inamdar [U3])

Let $\mathcal{I}$ *denote the collection of all* $\mathrm{cf}(\lambda)$*-indecomposable uniform ideals* I *over* λ^+ *for which* $\sup\{\theta \in \mathrm{Reg}(\lambda^+) \mid I \text{ is } \theta\text{-indecomposable}\} = \lambda$.
Then $\mathrm{unbounded}^{\square}(\mathcal{I}, \lambda^+)$ *holds. In particular, each* $I \in \mathcal{I}$ *is not weakly* λ^+*-saturated.*

Optimality

The first hypothesis cannot be dropped (Ben-David–Magidor, 1986).
The second hypothesis cannot be dropped either (take a strongly compact below λ).

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Then $\mathrm{unbounded}^{\square}(\mathcal{I}, \lambda^+)$ *holds. In particular, each* $I \in \mathcal{I}$ *is not weakly* λ^+*-saturated.*

Corollary (Kunen–Prikry, 1971, improving Chang 1967)

Suppose that I *is a* $\mathrm{cf}(\lambda)$*-indecomposable uniform ideal over* λ^+.
If I *is prime, then* λ *is singular and* $\sup\{\theta \in \mathrm{Reg}(\lambda) \mid I \text{ is } \theta\text{-indecomposable}\} < \lambda$.

Corollary (Eisworth, 2012)

Suppose that I *is a* $\mathrm{cf}(\lambda)$*-indecomposable uniform ideal over* λ^+ *for* λ *singular.*
If I *is weakly* λ*-saturated, then* $\sup\{\theta \in \mathrm{Reg}(\lambda) \mid I \text{ is } \theta\text{-indecomposable}\} < \lambda$.

Extensions of Hajnal's theorem and of [Sh:365, Claim 3.3]

Let κ be regular uncountable and $\mu \in [2, \kappa)$ be any cardinal.

Theorem (with Inamdar [U3])

Suppose I is a μ-complete uniform ideal over κ. Set $S := \{\alpha < \kappa \mid I \text{ is } \mathrm{cf}(\alpha)\text{-indec.}\}$.

1. *If there is a family of less than μ many stationary subsets of S that do not reflect simultaneously, then* $\mathrm{unbounded}^{\square}(I, \kappa)$ *holds.*

Corollary (Eisworth, 2010)

Suppose I is a μ-complete uniform ideal over κ, and let S be as above.
Consider $S' := \{\alpha \in S \setminus \mathrm{Reg}(\kappa) \mid I \text{ is weakly } cf(\alpha)\text{-saturated}\}$.
If there is a $\theta \in Reg(\kappa)$ such that I is weakly θ-saturated and θ-indecomposable then every family of less than μ many stationary subsets of S' reflects simultaneously.

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Theorem (with Inamdar [U3])

Suppose I is a μ-complete uniform ideal over κ. Set $S := \{\alpha < \kappa \mid I \text{ is } \mathrm{cf}(\alpha)\text{-indec.}\}$.

1. *If there is a family of less than μ many stationary subsets of S that do not reflect simultaneously, then* $\mathrm{unbounded}^{\square}(I, \kappa)$ *holds;*
2. *If there is a family of less than μ many stationary subsets of S that do not reflect simultaneously* *at regulars* *but* $\sup\{\theta \in \mathrm{Reg}(\kappa) \mid I \text{ is } \theta\text{-indecomposable}\} = \kappa$, *then* $\mathrm{unbounded}^{\square}(I, \kappa)$ *holds.*

Optimality: by our work with Lambie-Hanson and Zhang, the μ-completeness hypothesis cannot be dropped. In addition, Z. You announced that the first ω-Mahlo cardinal may carry an indecomposable ultrafilter.

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Theorem (with Inamdar [U3])

Suppose I is a μ-complete uniform ideal over κ. Set $S := \{\alpha < \kappa \mid I \text{ is } \mathrm{cf}(\alpha)\text{-indec.}\}$.

1. *If there is a family of less than μ many stationary subsets of S that do not reflect simultaneously, then* $\text{unbounded}^{\square}(I, \kappa)$ *holds;*
2. *If there is a family of less than μ many stationary subsets of S that do not reflect simultaneously at regulars but* $\sup\{\theta \in \mathrm{Reg}(\kappa) \mid I \text{ is } \theta\text{-indecomposable}\} = \kappa$*, then* $\text{unbounded}^{\square}(I, \kappa)$ *holds.*

Full statement of Corollary (with Inamdar [U3])

Suppose that κ is weakly inaccessible, $\chi \in \mathrm{Reg}(\kappa)$, and $E^{\kappa}_{\geq\chi}$ admits a stationary set that does not reflect at regulars. Then $\mathrm{Pr}_1(\kappa, \kappa, \kappa, \chi)$ holds.

This is sharp: by our work with Lambie-Hanson, $\mathrm{Pr}_1(\kappa, \kappa, \kappa, \chi^+)$ need not hold.

The end

Happy 80th birthday, Saharon!

מזל טוב בהגיעך לגבורות

Thank you for being a locksmith who opened countless doors for generations of mathematicians.