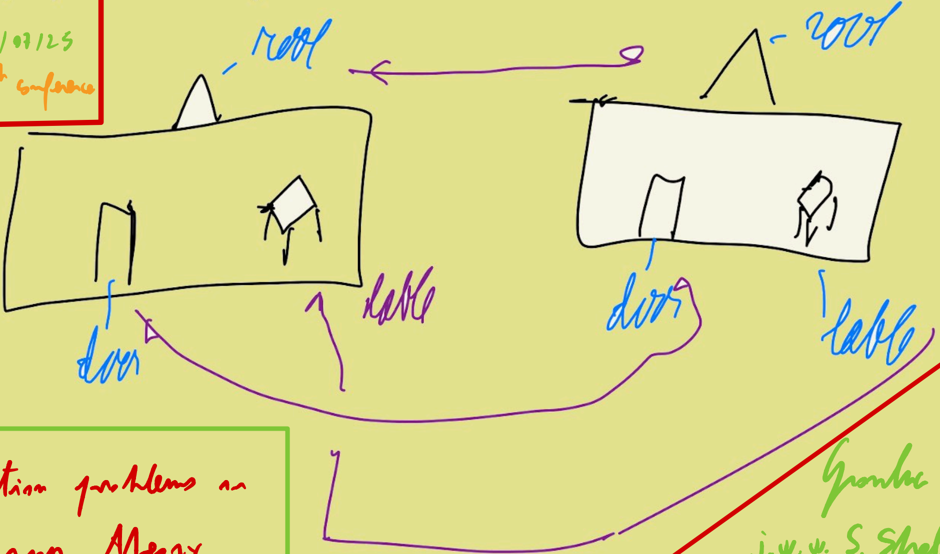
Vienna, 15/07/25

Shelah's 80th conference



Classification problems in abelian group theory

Gianluca Paolini

j.w.w. S. Shelah

As we all know Salvatore proved wonderful results in the most disparate fields of mathematics. In particular, he proved (and still proves each year) many beautiful results in abelian group theory. I thought about paying homage to him by overviewing our joint results in this area (abelian group theory).

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A BOREL REDUCIBILITY THEORY FOR CLASSES OF COUNTABLE STRUCTURES

HARVEY FRIEDMAN AND LEE STANLEY

In this paper a theory of Borel reducibility of isomorphism relations of $\mathcal{L}_{\omega_1\omega}$-theories of countable structures was proposed.

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We denote by Graphs_ω (resp. AB_ω) the Borel space of graphs (resp. abelian groups) with domain ω with the topology of "finite information".

$$\boxed{\begin{array}{l}\underline{\text{Question}}\ \exists \text{ Borel function } F : \mathrm{Graphs}_\omega \to \mathrm{AB}_\omega \\ \text{such that } \forall\, \Gamma, \Delta \in \mathrm{Graphs}_\omega \text{ we have} \\ \qquad \Gamma \cong \Delta \iff F(\Gamma) \cong F(\Delta)\ ?\end{array}}$$

Classes of str. s.t. ↑ happens are called Borel complete.

<u>Def.</u> A torsion-free abelian group is a subgroup of $\mathbb{Q}^{(n)}$ for $n \in \mathrm{Card}$. The rank of A is the smallest $n \in \mathrm{Card}$ s.t. $A \leq \mathbb{Q}^{(n)}$, e.g. $\mathrm{rk}(\mathbb{Z}^2) = 2$.

We denote by TFAb_ω the Borel space of torsion-free abelian groups with domain ω, and by TAb_ω the Borel space of torsion abelian groups with domain ω.

$(\forall a \in A \ \exists m < \omega \text{ s.t. } ma = 0)$

Fact (Friedman and Stanley) TAB_w is NOT Borel complete.

$\cong_n$ is iso relation on $A \in TFAB_w$ s.t. $rk(A) = n$.

Theorem (Thomas) $\cong_n <_B \cong_{n+1}$

[11] S. Thomas. *On the complexity of the classification problem for torsion-free abelian groups of rank two.* Acta Math. **189** (2002), no. 02, 287-305.

[12] S. Thomas. *The classification problem for torsion-free abelian groups of finite rank.* J. Amer. Math. Soc. **16** (2003), no. 01, 233-258.

What about $\cong$ on $TFAB_w$ of rank w?

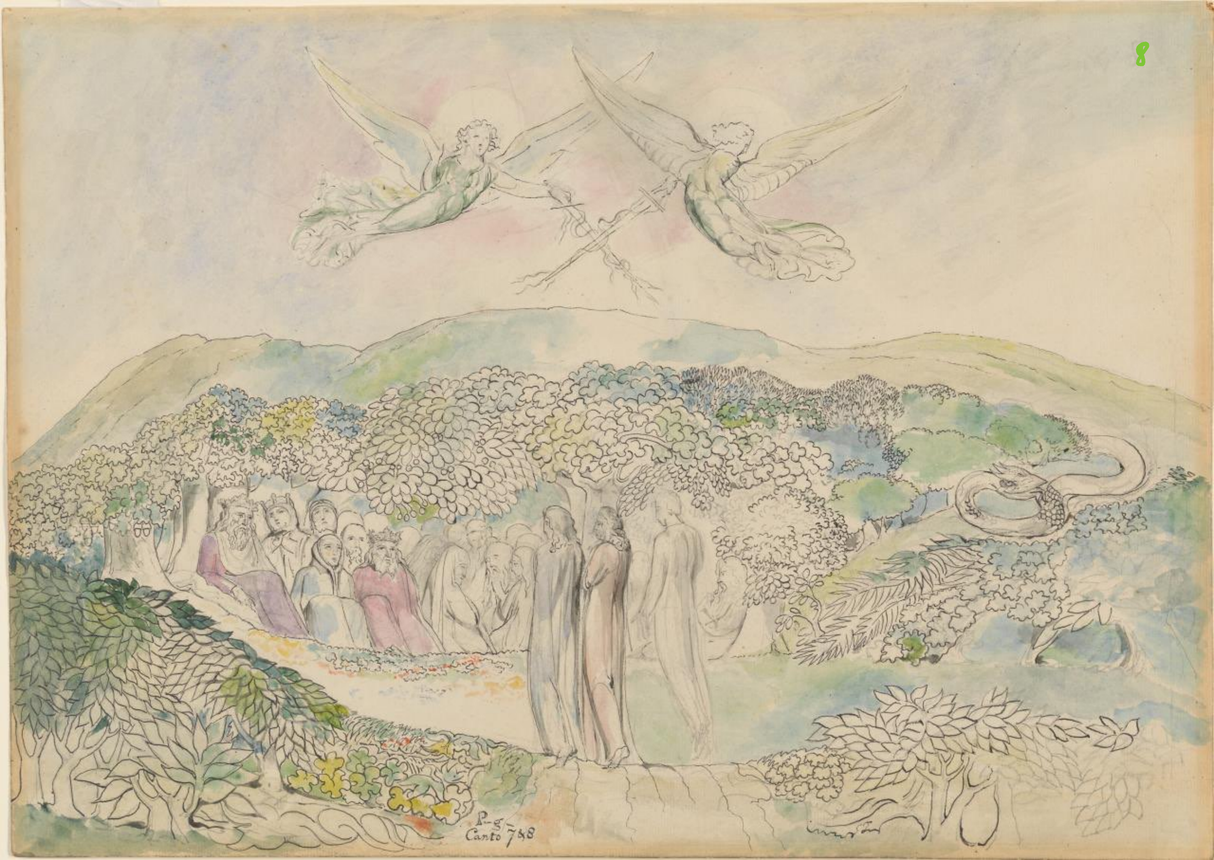
<u>Fact (Hjorth, 2002)</u> $\cong$ on $TFAB_\omega$ is NOT Borel.

<u>Fact (Downey + Montalbán, 2008)</u> $\cong$ on $TFAB_\omega$ is a complete analytic subset on $TFAB_\omega \times TFAB_\omega$.

A necessary but not sufficient condition for Borel complete.

Ma questa sonnolenza mi fu tolta
Subitamente da gente, che dopo
Le nostre spalle a noi era già volta.

Purgatorio, c. XVIII, v. 88-90.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Torsion-free abelian groups are Borel complete

By Gianluca Paolini and Saharon Shelah

Abstract

We prove that the Borel space of torsion-free abelian groups with domain ω is Borel complete, i.e., the isomorphism relation on this Borel space is as complicated as possible, as an isomorphism relation. This solves a long-standing open problem in descriptive set theory, which dates back to the seminal paper on Borel reducibility of Friedman and Stanley from 1989.

The solution of the problem first appeared on ArXiv in 2021.

Definition 1.3. *Let* $\mathbf{K}_\omega$ *be the Borel space of models with domain* ω *of a* $\mathfrak{L}_{\omega_1,\omega}$*-theory. The space* $\mathbf{K}_\omega$ *is said to be* faithfully *Borel complete if there is a Borel reduction* $\mathbf{F}$ *from* Graph_ω *(graphs with domain* ω*) into* $\mathbf{K}_\omega$ *such that for any invariant Borel subset* X *of* Graph_ω *the closure under isomorphism of the image of* X *under* $\mathbf{F}$ *is Borel.*

<u>Infinitary Vaught's conjecture</u> For every sentence φ in $L_{\omega_1,\omega}$ either φ has countably many countable models or continuum many countable models (up to isomorphism).

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Torsion-free abelian groups are faithfully Borel complete and pure embeddability is a complete analytic quasi-order

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Abstract In Paolini and Shelah (2024) we proved that the space of countable torsion-free abelian groups is Borel complete. In this paper we show that our construction from Paolini and Shelah (2024) satisfies several additional properties of interest. We deduce from this that countable torsion-free abelian groups are faithfully Borel complete, in fact, more strongly, we can $\mathfrak{L}_{\omega_1,\omega}$-interpret countable graphs in them. Secondly, we show that the relation of pure embeddability (equiv., elementary embeddability) among countable models of $\mathrm{Th}(\mathbb{Z}^{(\omega)})$ is a complete analytic quasi-order.

Abstract In Paolini and Shelah (2024) we proved that the space of countable torsion-free abelian groups is Borel complete. In this paper we show that our construction from Paolini and Shelah (2024) satisfies several additional properties of interest. We deduce from this that countable torsion-free abelian groups are faithfully Borel complete, in fact, more strongly, we can $\mathfrak{L}_{\omega_1,\omega}$-interpret countable graphs in them. Secondly, we show that the relation of pure embeddability (equiv., elementary embeddability) among countable models of $\mathrm{Th}(\mathbb{Z}^{(\omega)})$ is a complete analytic quasi-order.

Corollary Vaught conjecture reduces to the infinitary Vaught's conjecture for torsion-free abelian groups, i.e., if the infinitary Vaught's conjecture is true for torsion-free abelian groups, then the full (infinitary or not) Vaught's conjecture is true.

Recall that $\cong$ on Graphs_ω is NOT a complete analytic equivalence relation but bi-embeddability on Graphs_ω is a complete analytic equivalence relation!

In our construction we have:

$$\mathrm{Graphs}_\omega \qquad\qquad \mathrm{TFAB}_\omega$$

$$\Gamma \hookrightarrow \Delta \iff G_\Gamma \xhookrightarrow{\text{pure}} G_\Delta$$

where $A \leq B \in \mathrm{TFAB}$ is pure if $B \models m \mid a \Rightarrow A \models m \mid a$ for every $a \in A$ and $0 < m < \omega$.

Def. Let $A \in TFAB$, then we say that:

(1) A is endorigid if $End(A) \cong \mathbb{Z}$;

(2) A is Hopfian if $\forall f \in End(A)$ if f is surjective, then f is injective;

(3) A is rigid if $Aut(A) = \{id_A, -id_A\}$.

Theorem The three properties above are all complete co-analytic subsets of $TFAB_\omega$.

<u>Theorem</u> For all $\lambda \in \mathrm{Card}$ $\exists A \in \mathrm{TFAB}$ of size λ such that A is absolutely Whitehead, i.e., A is Whitehead and A remains Whitehead in every forcing extension of the universe.

This "effectivizes" a famous construction of Shelah of rigid abelian groups of size λ for every infinite λ.

It is an open problem to do the $\uparrow$ with "rigid"!!!

We now move to results on (non-Archimedean) Polish groups (not necessarily abelian).

<u>Def.</u> Let (G, τ) be a Polish group. Then G is said to be non-Archimedean if G has a neighbourhood base at the identity consisting of open subgroups.

Becker + Kechris (Groups for non-Archimedean)

$\downarrow$

<u>Question</u> Can an uncountable (non-Archimedean) Polish group be free (resp. free abelian)?

<u>Fact (Shelah)</u> No uncountable Polish group can be free (resp. free abelian).

The previous result was generalised as follows:

Paolini + Shelah, Polish topologies for graph products of groups, JLMS 100 (2019), no. 02, 383-403.

Corollary 10. *Let G be an abelian group which is a direct sum of countable groups, then G admits a Polish group topology if only if G admits a non-Archimedean Polish group topology if and only if there exists a countable $H \leqslant G$ and $1 \leqslant n < \omega$ such that:*

$$G = H \oplus \bigoplus_{\alpha<\lambda_\infty} \mathbb{Q} \oplus \bigoplus_{p^k|n} \bigoplus_{\alpha<\lambda_{(p,k)}} \mathbb{Z}_{p^k},$$

with λ_∞ and $\lambda_{(p,k)} \leqslant \aleph_0$ or $2^{\aleph_0}$.

(Here $\mathbb{Z}_{p^k}$ is simply the cyclic group $\mathbb{Z}/p^k\mathbb{Z}$.)

Another way of saying the above is that G has to be (eventually) a finite sum of finitely many vector spaces over $\mathbb{Q}$ or over a finite field.

In our paper we actually prove a MUCH MORE general results and in particular a characterization of the graph of groups $G(\Gamma, G_v)$ with all the G_v countable (or also covering free groups).

On the other hand, $\mathbb{Z}^\omega = \prod_{i\in\omega} \mathbb{Z}$ is non-Archimedean Polish and $\aleph_1$-free, i.e., every countable subgroup of $\mathbb{Z}^\omega$ is free!

We want to understand when such objects exist!

Definition 1.1. Let A be a torsion-free abelian group.

(1) We say that A is separable if every finite subset of A is contained in a free direct summand of A.
(2) We say that A is torsionless if for every $0 \neq a \in A$ there is $f \in \mathrm{Hom}(A, \mathbb{Z})$ such that $f(a) \neq 0$.
(3) We A is $\mathbb{Z}$-homogenenous if every element has type $\mathbf{0}$ (cf. 3.7).

(this means that $\forall a \in A$, modulo dividing by some n, a is not divisible by any $m \in \mathbb{N}$ (like $1 \in \mathbb{Z}$).)

$$\text{separable} \Rightarrow \text{torsionless} \Rightarrow \aleph_1\text{-free} \Rightarrow \mathbb{Z}\text{-homogenenous}.$$

So we have $\mathbb{Z}^\omega$ is separable and non-Archimedean Polish

separable $\Rightarrow$ torsionless $\Rightarrow$ $\aleph_1$-free $\Rightarrow$ $\mathbb{Z}$-homogenenous.

Furthermore, there cannot be an uncountable Polish free abelian group.

<u>Question</u> Are there other uncountable separable non-Archimedean Polish abelian groups???

Theorem 1.2. *There are continuum many separable (hence torsionless, hence $\aleph_1$-free) abelian non-Archimedean Polish groups which are not topologically isomorphic to product groups and are pairwise not continuous homomorphic images of each other. Furthermore, all these groups can be taken to be inverse limits of torsion-free completely decomposable groups (i.e., direct sums of* TFAB *of rank* 1*).*

The construction uses what we call "the engine", i.e., inverse systems of trees of completely decomposable TFABs, indexed by subsets of the prime numbers.

Notation 2.2. Let X be a Polish space. The Effros structure on X is the Borel space consisting of the family $\mathcal{F}(X)$ of closed subsets of X together with the σ-algebra generated by the following sets $\mathcal{C}_U$, where, for $U \subseteq X$ open, we let:

$$\mathcal{C}_U = \{D \in \mathcal{F}(X) : D \cap U \neq \emptyset\}.$$

Fact 2.6. The closed subgroups of S_∞ form a Borel subset of $\mathcal{F}(S_\infty)$ (see [9]), which we denote by $\mathrm{Sgp}(S_\infty)$, together with the Borel structure inherited from $\mathcal{F}(S_\infty)$.

Using this we can investigate the Borel complexity of subsets of $\mathrm{Sgp}(S_\infty)$ or equivalence relations on it.

Theorem 1.3. *Determining if a non-Archimedean Polish group is in $\mathcal{C}$ is a complete co-analytic problem in the space of closed subgroups of S_∞ for the following classes $\mathcal{C}$ of abelian groups:*

(1) $\mathbb{Z}$-homogeneous;
(2) $\aleph_1$-free;
(3) torsionless;
(4) separable.

THANK YOU FOR YOUR ATTENTION